Branko Grünbaum

My favorite conjectures

'Tis better to have conjectured and failed
Than never to have conjectured at all.
(With apologies to Alfred, Lord Tennyson)

Introduction.

Many people include a variety of open problems in their writings. This helps advance knowledge by presenting challenges to the readers, as well as giving them ideas regarding possible directions of extension of the available results. The positive aspects of such questions are greatly enhanced if the open problems are formulated as explicitly stated conjectures.

My inclination is — and was in the past — to state conjectures whenever it seems that I have a reasonable understanding of the situation, even though I lack the ability to decide the validity of the claim. Over the years, this led me to make many conjectures; I admit that some were rather outrageous in their generality. These conjectures experienced varied fates: some were proved, others demolished, and still others ignored. In this talk some of the conjectures close to my heart will be discussed, regardless of the outcome. So here we go, without any particular order, but starting with three easily stated conjectures which had very different fates.

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1. In G-1958 I made the following conjecture:

Conjecture 1. If every five members of a family of disjoint translates of a compact convex set in the plane have a common transversal, then there exists a common transversal for all members of the family.

I am happy to report that thirty years later our friend Helge Tverberg established the validity of this conjecture in 1989.

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2. In contrast, a more recent conjecture on polygons in the plane met a quick rebutal. For a given quadrangle, there are seven different (mutually exclusive) possibilities regarding the lengths of its sides:

(1) all sides are equal;
(2) three sides are equal, different from the fourth;
(3) two pairs of adjacent sides are equal;
(4) two pairs of opposite sides are equal;
(5) one pair of adjacent sides are equal, the other two are different from these and from each other;
(6) one pair of opposite sides are equal, the other two are different from these and from each other;
(7) all four sides are different.

Completely analogous seven possibilities arise with respect to the angles, and now one can ask: *Which of the 49 pairs of conditions can be realized by a convex quadrangle?*

The answer is that *precisely twenty* of the 49 possibilities can be realized. Specifically, the table below shows which are the pairs that correspond to quadrangles, and Figure 1 shows representatives of these types (cued to the letters in the table).

Side type	Angle type (1)	(2)	(3)	(4)	(5)	(6)	(7)
(1)	A			B			
(2)			C		D		E
(3)		F				G	
(4)	H			J			
(5)		K			L	M	N
(6)			P		Q		R
(7)		S			T	U	V

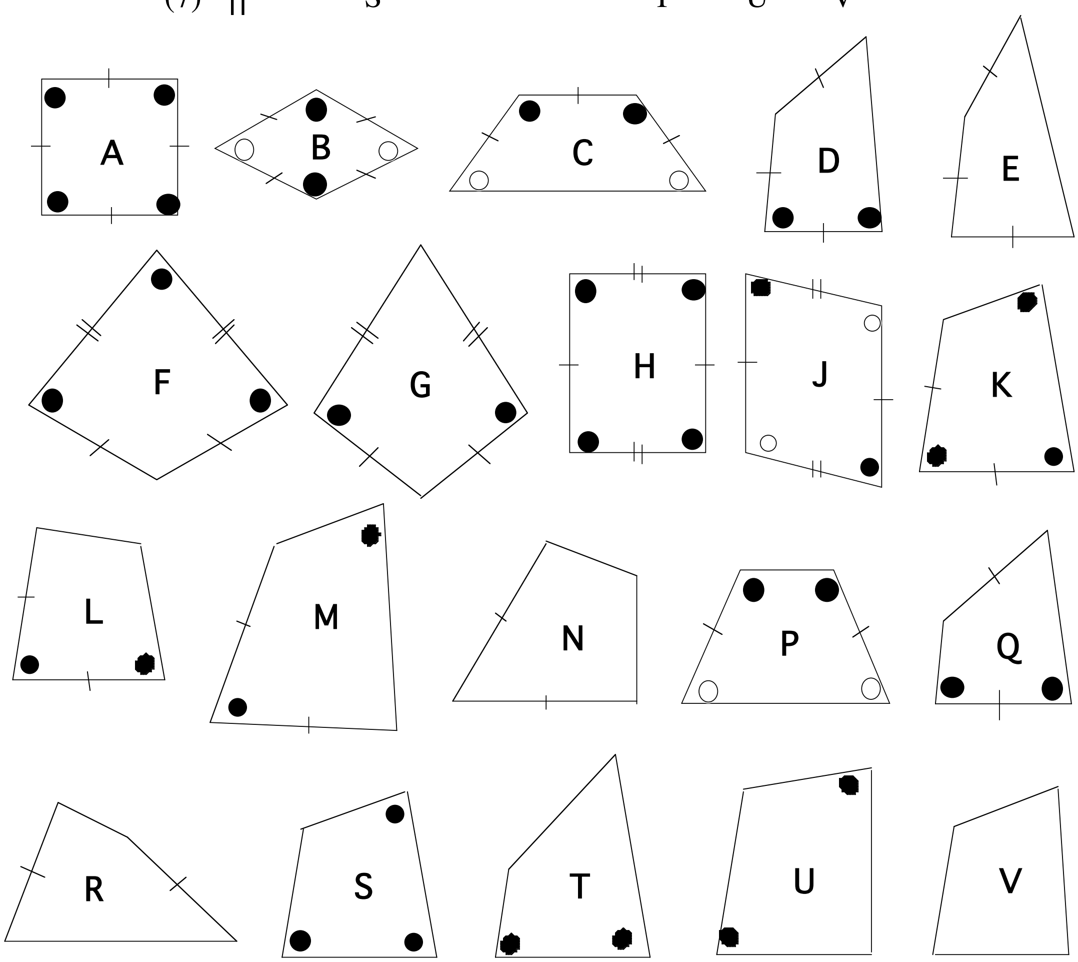

Figure 1.

Now, one look at the table yields the startling conclusion: *There is complete reciprocity between the sides and the angles*, as is evident in the symmetry with respect to the main diagonal of the entries in the table. This led me to suggest in G-1995:

Conjecture 2. For every $n \geq 3$, there is a reciprocity of a similar kind for the possible relations among sides and angles of convex n-gons.

The conjecture clearly holds for triangles, where there are just the traditional three possibilities. I did not (and do not) know whether the conjecture is valid for $n = 5$, or the next few values. However, Auroux [1996] showed by an example (see Figure 2) that for $n = 12$ the conjecture fails, and conjectured that it is the smallest value with this property.

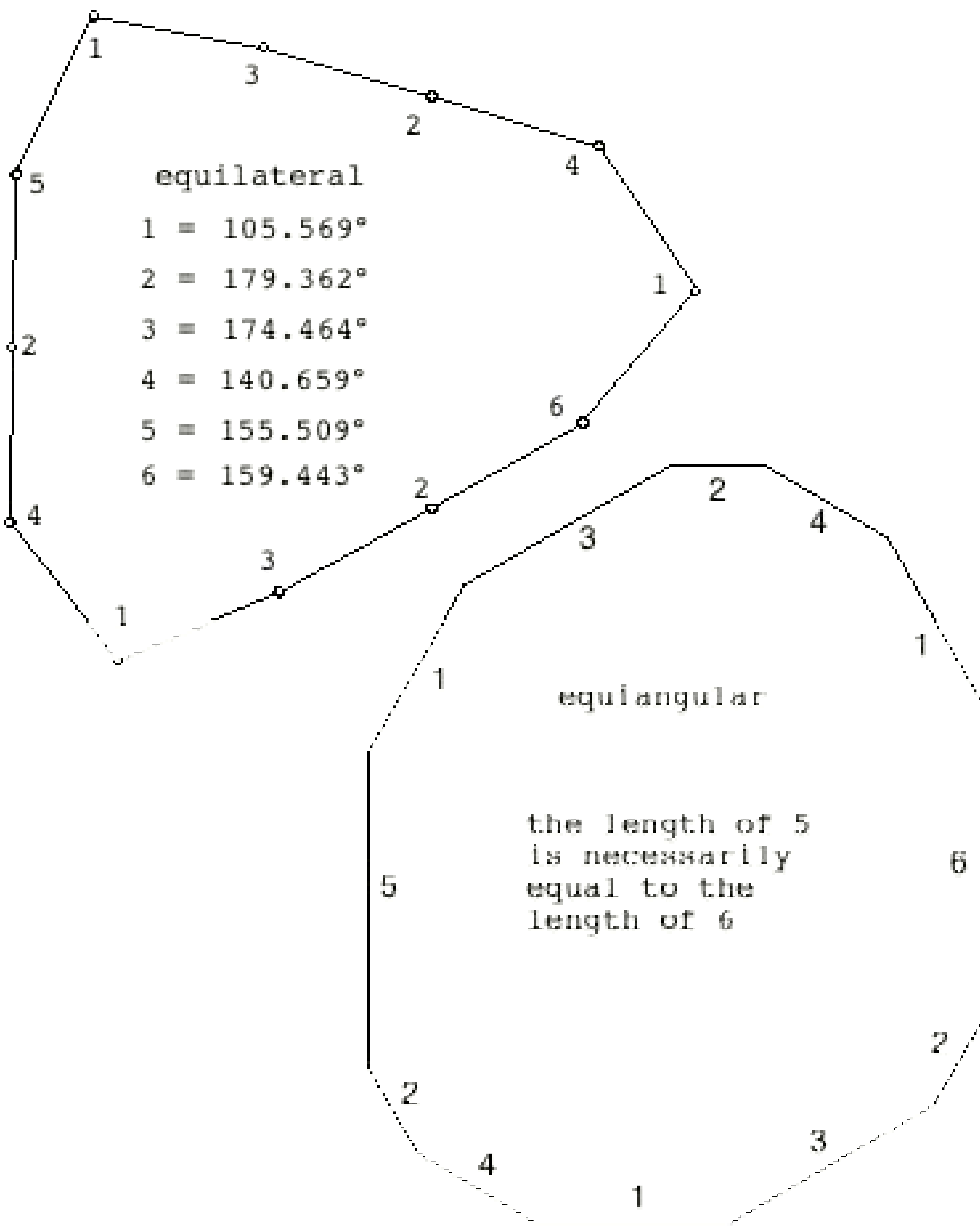

Figure 2.

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3. Similarly split was the fate of another old conjecture. If K is a convex body in E^d, and S is a subset of the boundary of K, we say that S is an **inner illuminating set** for K if every boundary point of K can be seen from some point of S via a segment through the interior of K. An inner illuminating set is **primitive** if no proper subset is an inner illuminating set. In Figure 3, the vertices of the pentagon are an inner illuminating set which is not minimal. Any triplet of these vertices, not all consecutive, is a minimal inner illuminating set. It is easy to prove that every convex body in the plane admits a primitive inner illuminating set of at most 4 points, and that this is best possible: the vertices of any quadrangle are such a set. This led me to conjecture in G-1965 the following general statement:

Conjecture 3. The maximal number of points in a primitive inner illuminating set of a d-dimensional convex body is 2^d.

This conjecture was established for $d = 3$ by Soltan [1995]. However, in a recent collaboration of Soltan with two of our colleagues it was proved that this is the limit of the validity of my conjecture. Boltyanski, Martini and Soltan [1999] have shown that *for every $d \geq 4$ there exists a d-dimensional convex body such that for any positive integer $n \geq 2$ it admits a primitive inner illuminating set of at least n points*. It should be mentioned that they also establish that no comvex body admits an infinite primitive inner illuminating set.

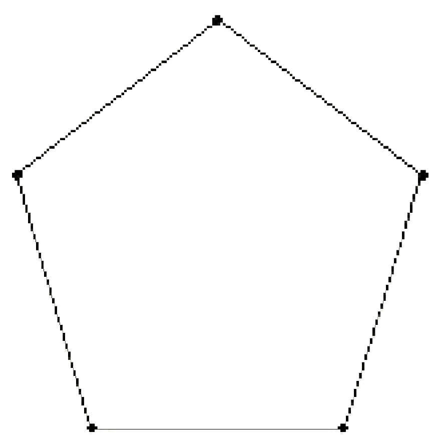

Figure 3.

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4. Many interesting questions are related to **spanning trees** in the graphs of convex 3-polytopes. If P_1 and P_2 form a pair of dual polytopes of this kind then the edges of P_1 and P_2 are in a canonical correspondence. The edges of any spanning tree in the graph of P_1 correspond in the graph of the dual polytope P_2 to the complement of a spanning tree, which we call **complementary** to the starting tree. An illustration of a complementary pair of spanning trees is given in Figure 4. It is interesting that the complementarity of the spanning trees of the cube and the octahedron was observed by Jeger [1975] without noticing that it is a general property of dual polytopes.

Barnette [1966] proved that the graph of vertices and edges of every 3-dimensional convex polytope contains a spanning tree of maximal valence 3; we shall call such trees **Barnette trees**. This led me to the following:

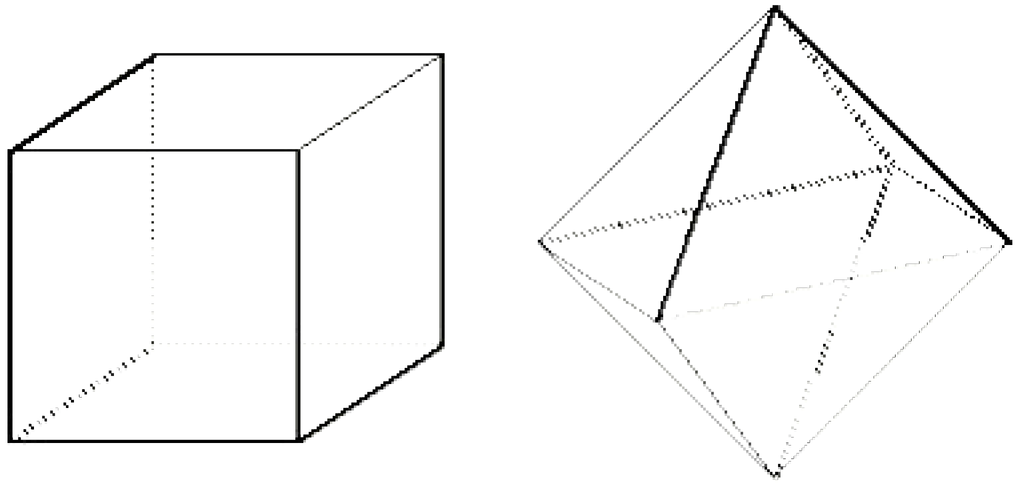


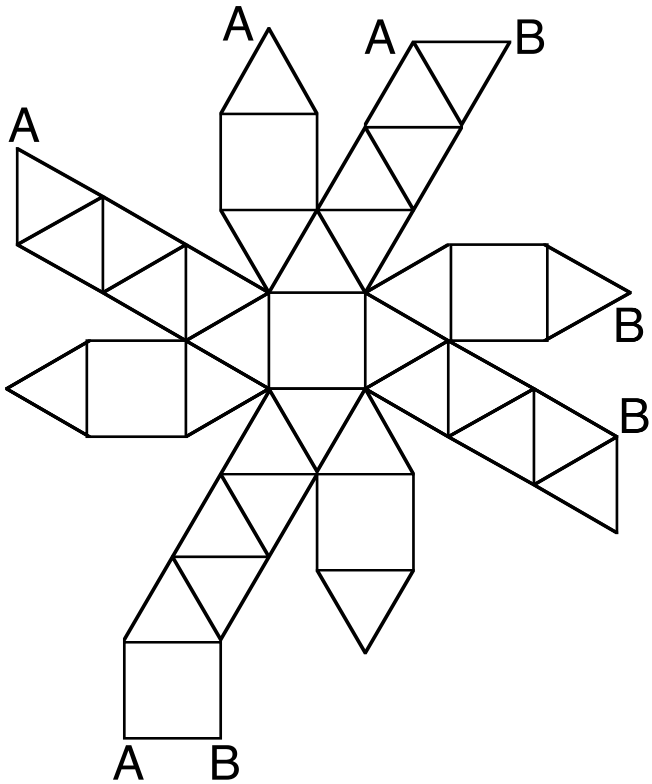
Figure 4.

Conjecture 4. (Grünbaum [1970], p. 1147/1148) In every 3-dimensional convex polytope there exists a complementary pair of Barnette trees.

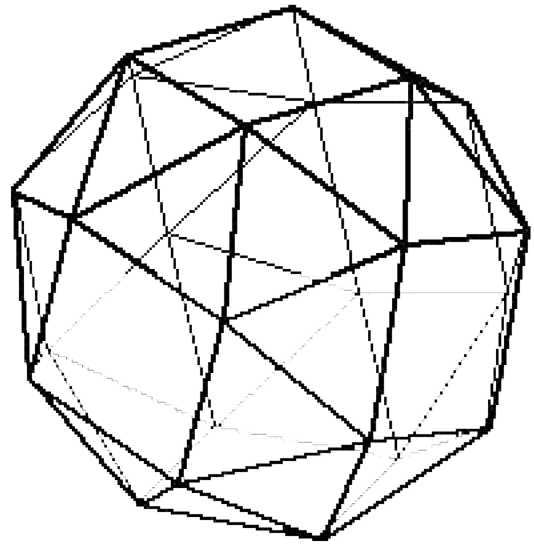
Clearly, Barnette's result implies the validity of Conjecture 4 for all simplicial 3-polytopes, as well as for all simple ones.

Each spanning tree T of a convex polytope P determines a **face-tree** \(T) of P. By face-tree we understand the *combinatorial* object analogous to the well-known planar nets often used in the construction of polytopes from cardboard; it is clearly isomorphic to the complementary tree of T. This may sound a bit imprecise for a definition, but it should be sufficient to indicate what I have in mind — namely the combinatorial structure underlying a net, which is meaningful even if the net does not exist because the face-tree cannot be embedded in the plane on account of overlaps. Face-trees and nets can be used to illustrate spanning trees since the boundary of the face-tree \(T) results by slitting the boundary-complex of P along the edges of a spanning tree T, and, for a net, unfolding the resulting set of polygons into a plane. Figure 5 illustrates in part (a) a face-tree (in fact, a net) of the Archimedean "snub cube" shown in part (b). A spanning tree of the snub cube, complementary to this face-tree, is shouwn in part (c). This particular net of the snub cube goes back to Dürer [1525], who, by the way, seems to be the first to have devised nets for convex 3-polytopes. Dürer's net does not represent a Barnette tree of the snub cube, since the spanning tree is 4-valent at the vertices marked A and B. However, by minor rearrangements a Barnette tree can be obtained, as shown in Figure 6. Representations of spanning trees by nets can also be used in a very simple way to check the validity of Conjecture 4 in specific cases, since the complementary tree in the dual is, at the same time, a face-tree of the original polytope. In Dürer's net shown in Figure 5(a) the complementary spanning tree is also not a Barnette tree: the central square has four attached faces, hence corresponds to a 4-valent vertex of the complementary tree. However, the tree represented by Figure 6 clearly has a complementary tree that is a Barnette tree, and so provides a not completely trivial illustration of Conjecture 4.

While Conjecture 4 is combinatorial in character, spanning trees and the corresponding unfolding of the boundary complexes of 3-polytopes lead to many metric questions as well. A very old one was posed by Gergonne [1818], who asked for a characterization of those planar polygons which admit a partition that is a net for some convex 3-polytope. This received an answer of sorts in the work of Alexandrow [1955], although not in quite the sense that Gergonne probably meant. However, it is noteworthy



(a)



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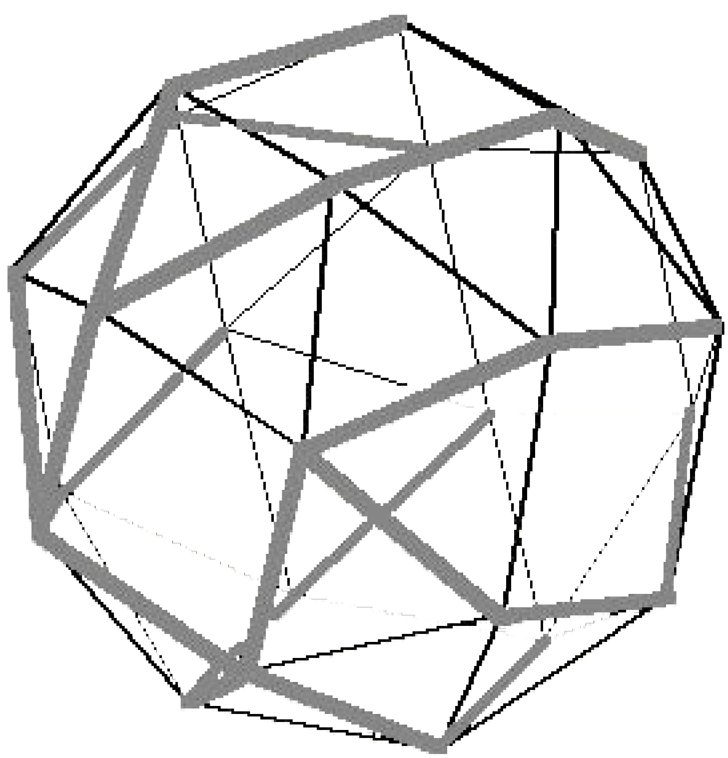
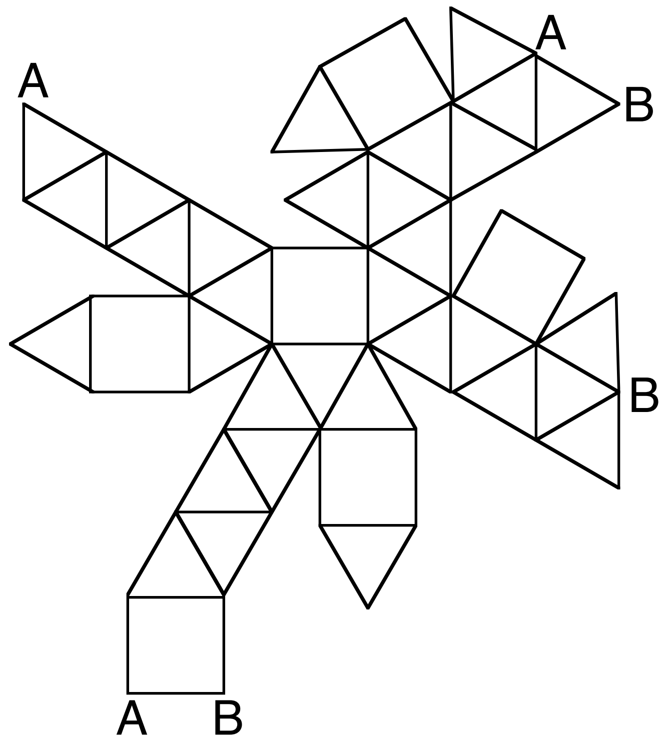
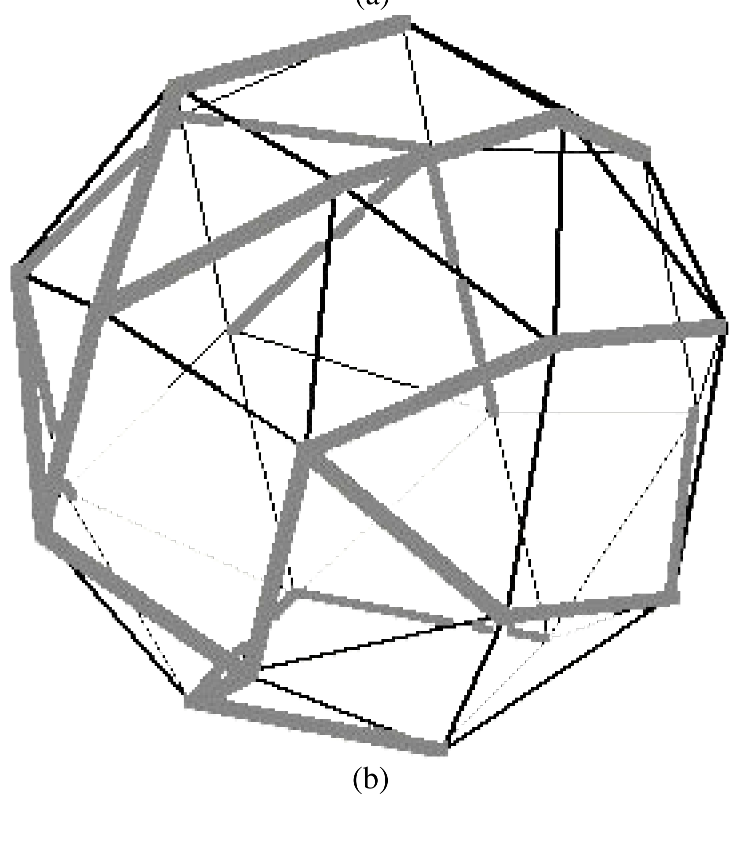


(c)

Figure 5.



(a)



(b)

Figure 6.

that Gergonne introduced the question by stating: "Every convex polyhedron has as a plane net a convex or nonconvex polygon, subdivided into polygonal compartments".

This reflects the naïve spirit of early 19th century -- jumping to conclusions based on the outcome in very simple situations. It seems that the question of existence of nets of arbitrary convex 3-polytopes was explicitly posed only much later, by Shephard [1975]; he examined several related problems as well, and provided most of the few known answers to some of them. The question was repeated by Croft et al. [1991] and Ziegler [1995], together with many related problems. It is unresolved to this day, and led me to formulate

Conjecture 5. (Grünbaum [1991]). Every convex 3-polytope has a net.

Not every spanning tree of a 3-polytope leads to a net: it is possible that the slitting along the edges of a spanning tree and subsequent unfolding results in a family of polygons that partially overlap. An example (from Grünbaum [1991]) of a face-tree which is not a net is indicated in Figure 7.

A much weaker version of Conjecture 5 is that every polytope is combinatorially equivalent to a polytope with a net.

On the other hand, for some polytopes every spanning tree corresponds to a net. All tetrahedra, as well as polyhedra combinatorially equivalent to the 3-sided prism, have this property. The assertion in Schlickenrieder [1997] that there is a tetrahedron (due to M. Namiki) for which one spanning tree leads to a face-tree which is not a net is erroneous. It is not known how many combinatorial types of 3-polytopes share with tetrahedra and 3-prisms the property that all trees on every polytope of this type lead to nets. We venture the following

Conjecture 6. Every combinatorial type of 3-polytopes admits a representative for which all spanning trees yield nets.

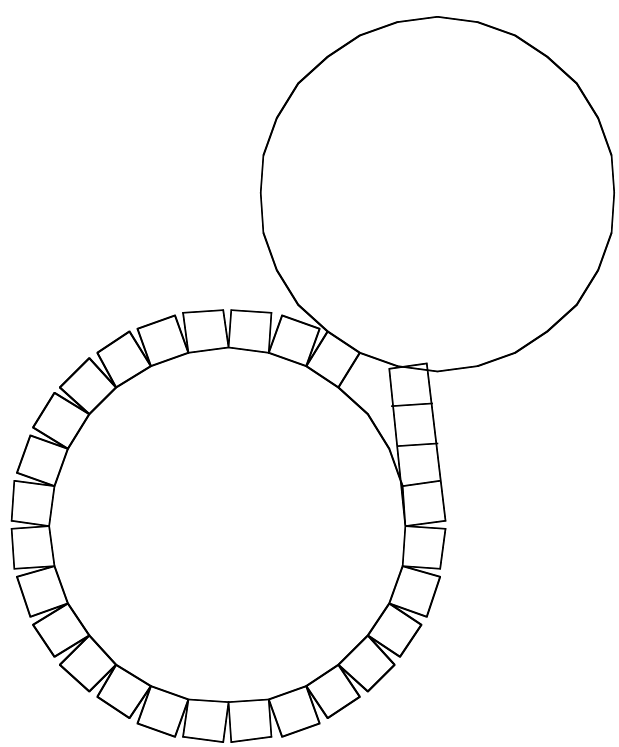


Figure 7.

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5. By **acoptic** polyhedron I mean a polyhedron in 3-space which consists of a family of simple planar polygons (**faces**) that satisfy a number of conditions. Specifically:

(i) pairs of faces intersect only along **edges** or **vertices** common to these faces;

(ii) the faces incident with a vertex form only a single circuit;

(iii) all faces are strongly connected to each other.

Thus, acoptic polyhedra are cell decompositions of embedded 2-manifolds. They greatly generalize the boundary complexes of convex 3-polytopes, and it is remarkable how little is known about such a relatively simple family of objects.

The extent to which properties of acoptic polyhedra may differ from those of convex polytopes is possibly best illustrated by the contrast between Cauchy's rigidity theorem (Cauchy [1813]), and the existence of flexing triangulations of the sphere (Connelly [1978]). (A recent development is the proof by R. Connelly, I Sabitov and A. Walz [2000] of the "bellows conjecture", that all flexing polyhedra enclose a constant volume.) For other facts concerning acoptic polyhedra see Grünbaum [1998]. Here I would like to discuss a problem about acoptic polyhedra that has a relatively extensive history:

Which cell complexes can be realized by acoptic polyhedra?

The first result in this direction is the famous theorem of Steinitz [1922] (see in particular Steinitz and Rademacher [1934]), characterizing cell complexes that are combinatorially equivalent to the boundary complexes of convex 3-polytopes. (See Grünbaum [1967], Barnette and Grünbaum [1969], Ziegler [1995] for more accessible treatments; avoid Lyusternik [1956] as the proof given there is inadequate and distorts Steinitz's achievement.) In modern terminology, this is usually stated in the form: A graph G is isomorphic to the graph of a convex 3-polytope if and only if it is planar and 3-connected. However, even the simplest case beyond this is completely open, despite many attempts over several decades. This can be formulated as asking whether every **triangulation of the torus** is combinatorially equivalent to an acoptic polyhedron. For the earliest mention see Duke [1970], for details about the history of this still unsolved problem and several of its relatives see Grünbaum [1998]. Naturally, this has to be understood as restricted to **proper triangulations**, that is, those in which no two edges share the same vertices. The decomposition of the torus into triangular regions, shown in Figure 8, is clearly not such a triangulation. It is obvious that it cannot be realized by an acoptic polyhedron, since it has two vertices that are joined by two distinct edges. From now on we shall assume that no two vertices determine more than one edge.

On the other hand, acoptic polyhedra may have **overarching elements**, and thus be only "generalized complexes". Here we say that a (generalized) cell complex or polyhedron has *overarching elements* if it contains two vertices and two faces that are mutually incident but are not all incident with one edge. Examples of such polyhedra are shown in Figure 9.

However, not every cell-decomposition of an orientable 2-manifold is combinatorially equivalent to an acoptic polyhedron. It seems that the presence of too many sets of overarching elements may preclude the representation of a manifold by an acoptic polyhedron. Here is an example.

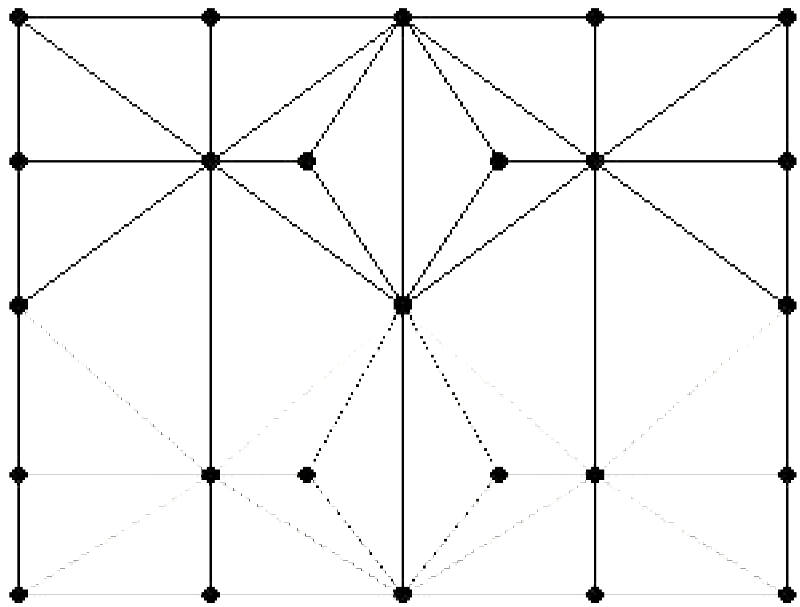

Figure 8.

We start the construction of the closed manifold M by taking ten distinct points A, B, C, D, E, F, G, H, J, K (which will be some of the vertices of M), and nine triplets of cells. Each triplet of cells should contain one of the following sets of named vertices on its boundary, but be otherwise disjoint; it is convenient to think of the cells of each triplet as being close to each other, and having corresponding distinct sets of additional vertices which we need not name. The sets of named vertices are {A, B, G}, {A, C, E}, {A, F, K}, {B, C, H}, {B, D, F}, {C, D, J}, {D, E, G}, {E, F, H}, {G, H, J, K}. An illustration of the cells incident with each vertex is shown schematically in Figure 10. Only one cell of each triplet is shown; the other two should be understood as being close to the one shown, and the dotted lines indicate the connections between the three members of each triplet. The 27 cells may best be imagined as being piecewise linear, and as having a definite orientation. This can be visualized by coloring red one side of each, and green the other side. This yields a (generalized) complex which is an orientable manifold N with boundary; we consider one of the sides of N colored red, the other green.

As the next step in the construction of M we take a manifold with boundary N' (very) close to N on its red side, and color red the side of N' facing N. Finally, we join corresponding closed circuits on the boundaries of N and N' by suitable ribbons, and obtain the desired orientable 2-manifold M. It is easy to verify that M has genus 15. Using any cell-complex decomposition of M which keeps the original 27 cells intact we obtain a complex which we claim is not combinatorially equivalent to any acoptic polyhedron.

Indeed, let's assume that an acoptic polyhedron P combinatorially equivalent to M exists . If a pair of faces of P have at least three vertices in common, then the faces are either coplanar, or else all these vertices are collinear. Since in M there are triplets of faces with three or four vertices in common, the first eventuality cannot apply to all pairs, and hence all vertices common to such a triplet of faces are collinear. It follows that each of the nine sets of vertices listed at the start of the construction determines a line. In particular, the eight triplets determine a (possibly skew) hexagon ABCDEF

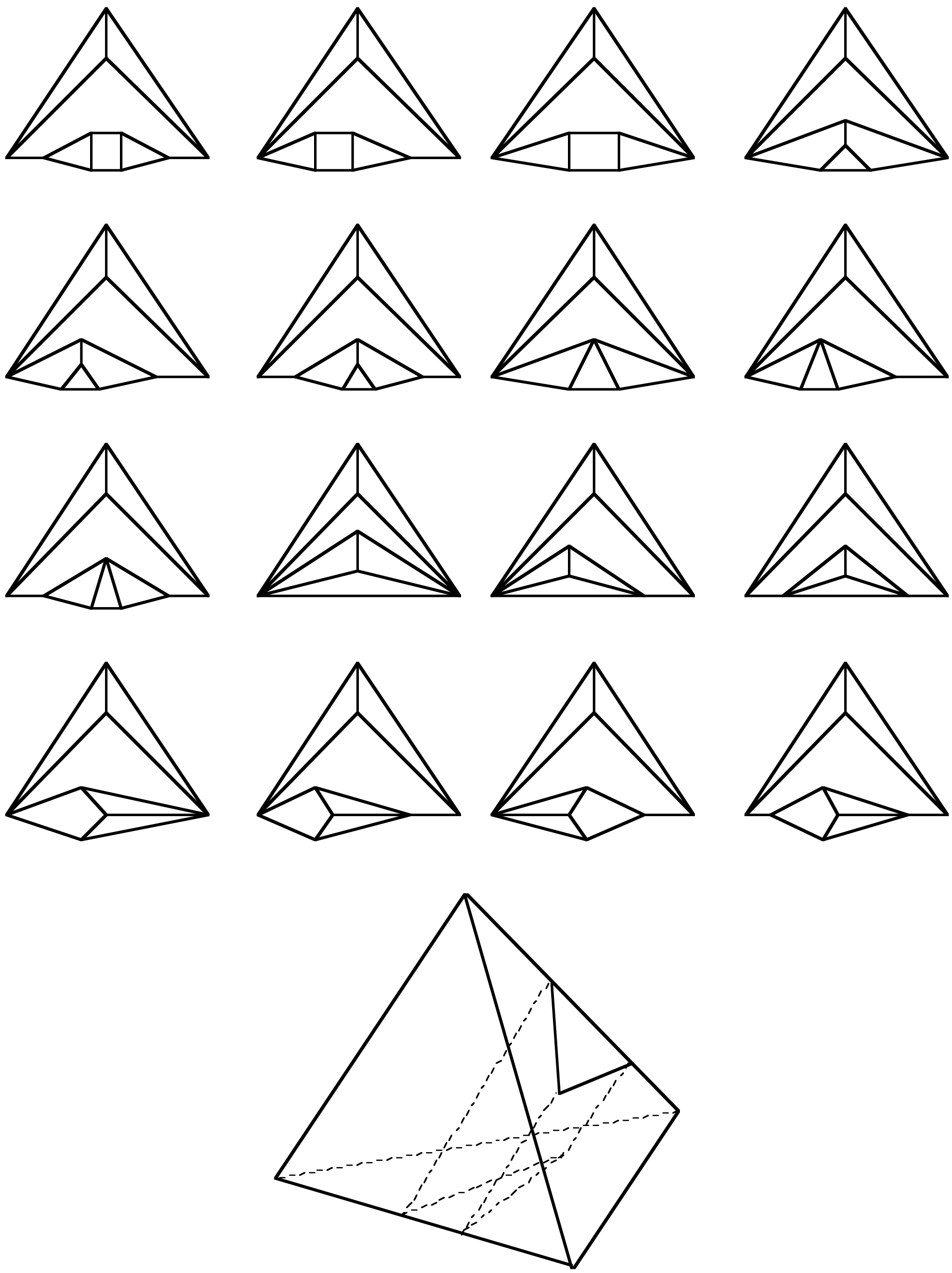

Figure 9.

inscribed in lines ACE and BDF, see Figure 11. By the Pappus-Pascal theorem, the line GH contains a point at which it is met by both the line CD and the line AF; thus the points J and K must coincide, contrary to the starting assumption that all ten named points are distinct.

It seems that simpler manifolds, more familiar in their presentation as selfintersecting polyhedra, may fail to be realizable by acoptic polyhedra. Specifically, for the Kepler-Poinsot regular polyhedra $\{5,5/2\}$ and $\{5/2,5\}$, which have many sets of overarching elements, no acoptic realizations of the underlying cell complexes seem to be known. I conjecture that no such realization exists. Unexpectedly, it seems that a confirmation of this conjecture is quite elusive. However, it the opposite direction I formulated in [G-1998a] the following, which includes various earlier conjectures:

Conjecture 7. (**General Realizability Conjecture.**) A cell-complex decomposition of a compact orientable 2-manifold M is realizable by an acoptic polyhedron provided M contains no overarching elements.

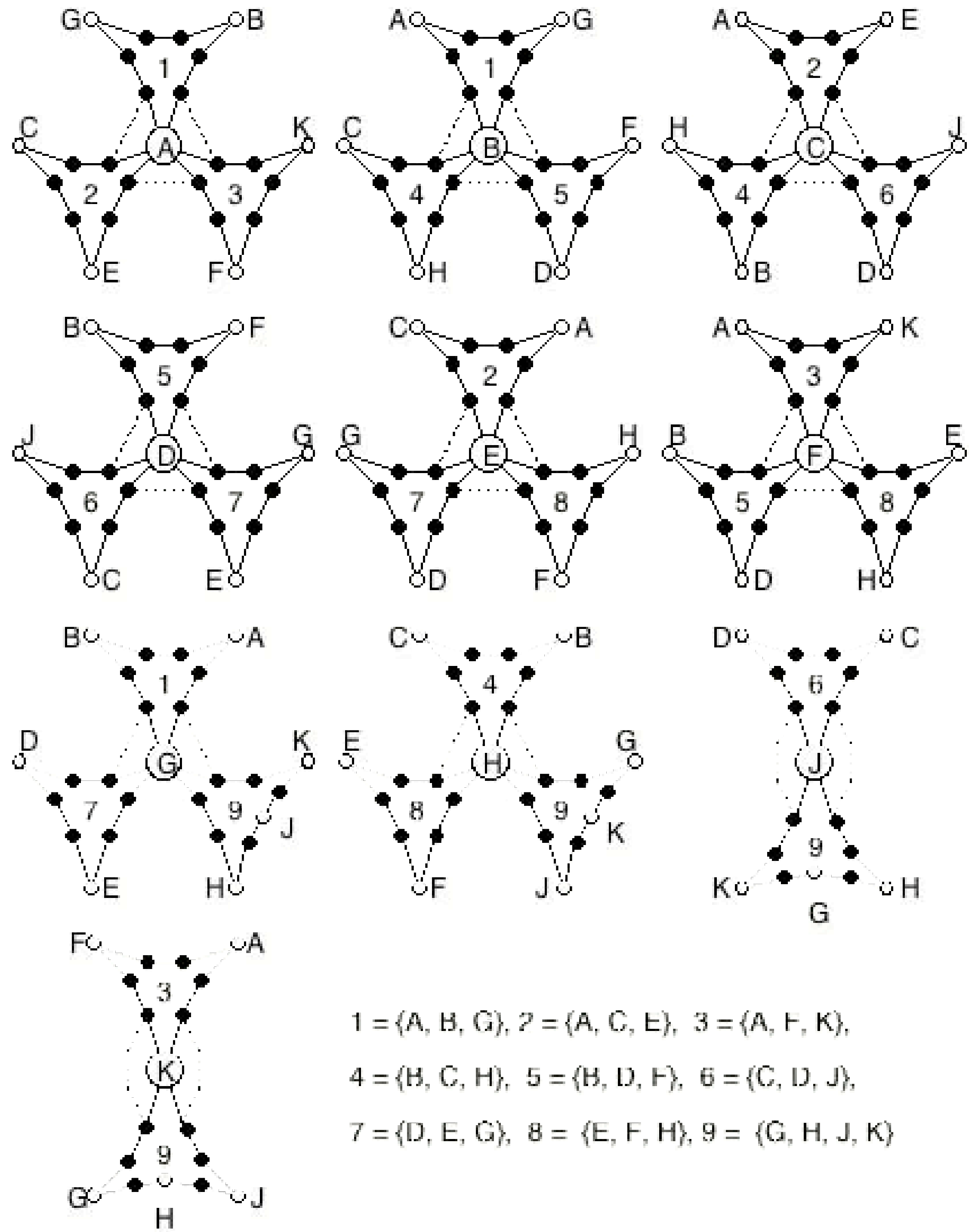

Figure 10.

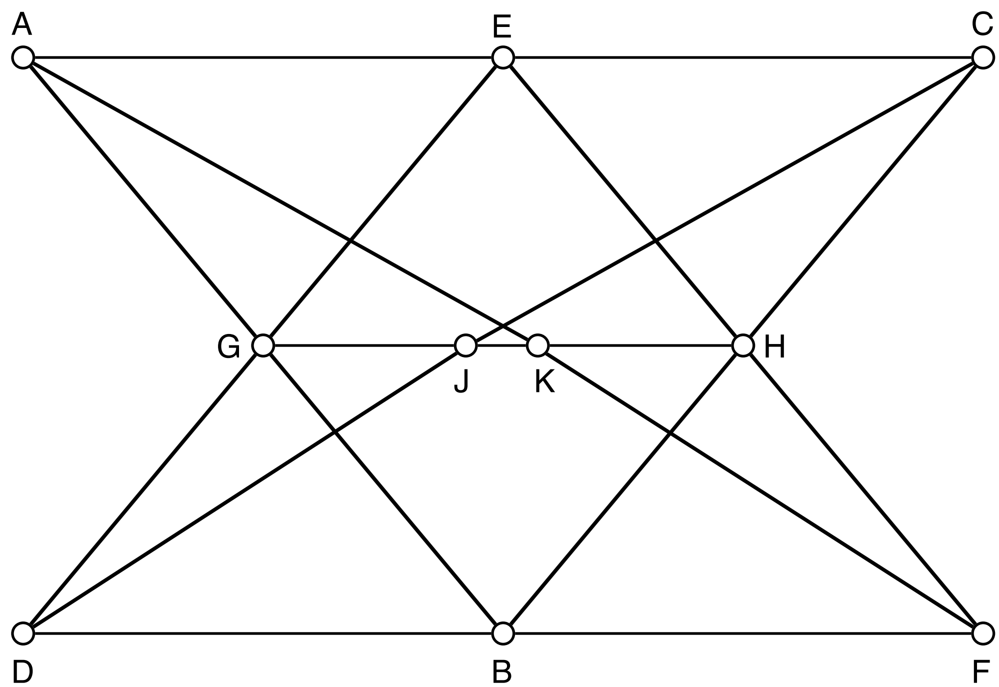

Figure 11.

The higher-dimensional analogs of acoptic polyhedra seem to have been even more neglected. For example, let us call **proper triangulation** any decomposition of the 3-sphere into topological simplices such that the link of each simplex is a sphere of the appropriate dimension. The following seems to be unresolved:

Conjecture 8. Every proper triangulation of the 3-sphere is combinatorially equivalent to a realization by geometric simplices in which there are no intersections of non-incident simplices.

Probably an analogous conjecture holds in all dimensions.

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6. It is well known that the validity of the Four-Color Theorem for maps on the sphere is equivalent to the possibility of properly 3-coloring the edges of every simple (3-valent) map, or the edges of every triangulation. Coloring the countries on maps on other manifolds requires larger number of colors. However, in contrast to this, many years ago I conjectured [G-1969] that the situation is different when triangulations are considered.

Conjecture 9. Every triangulation of each orientable 2-manifold admits a proper 3-coloring of its edges.

As (slightly) supporting evidence for this conjecture, in Figure 12 are shown proper edge 3-colorings of the 21 irreducible triangulations of the torus. These are precisely those proper triangulations of the torus in which no edge can be contracted; they have been studied in connection with the problem of realizability of triangulated tori by acoptic polyhedra.

No counterexample seems to be known for the even more general

Conjecture 10. Every (proper) triangulation of each orientable d-dimenional manifold admits a coloring of its (d-1)-dimensional faces by d+1 colors, in such a way that the (d-1)-dimensional faces of each d-simplex have all the different colors.

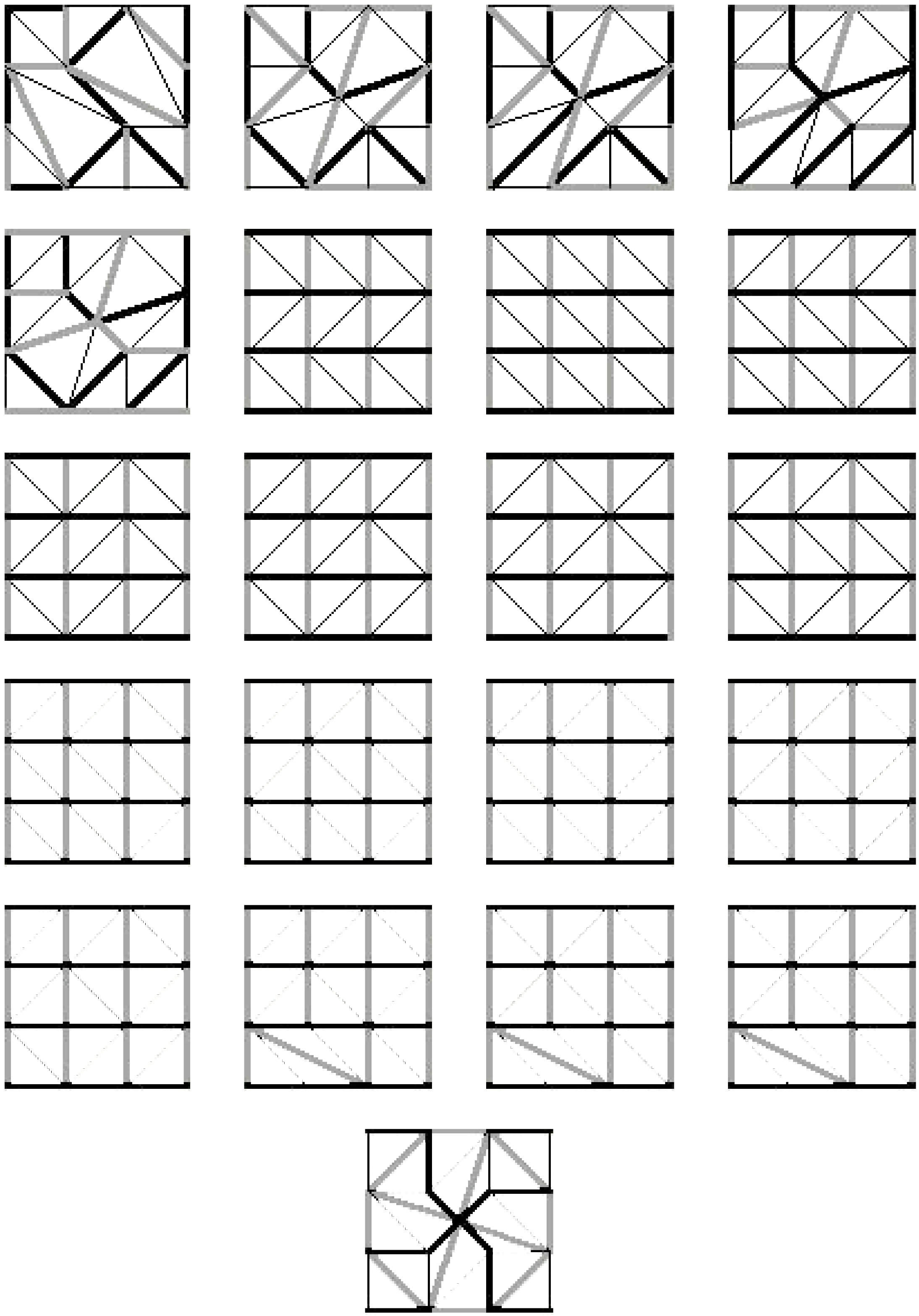

Figure 12.

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7. The cell-decomposition of the projective plane determined by any given finite family of (straight) lines is called an **arrangement.** Many interesting properties of arrangements are known, but many questions are still open. An arrangement is said to be **simplicial** if all the cells are simplices. The class of simplicial arrangements was first defined by Melchior [1940]; he also posed the problem of finding all such arrangements. Examples of simplicial arrangements are shown by their presentation in the extended Euclidean plane in Figure 13. There are three infinite families of such arrangements: The **near-pencils**, the arrangements determined by regular polygons and their lines of symmetry, and the ones obtained by adding the line at infinity to arrangements of the previous type, based on even-sided regular polygons. The arrangements in the left column in Figure 13 include the line at infinity, those in the middle may include the line at infinity and then yield members of the third family.

There is a large number of other simplicial arrangements; a few examples of such **sporadic** simplicial arrangements are shown in Figure 14. As a more specific version of earlier conjectures, the following formulation was proposed in GS-1984:

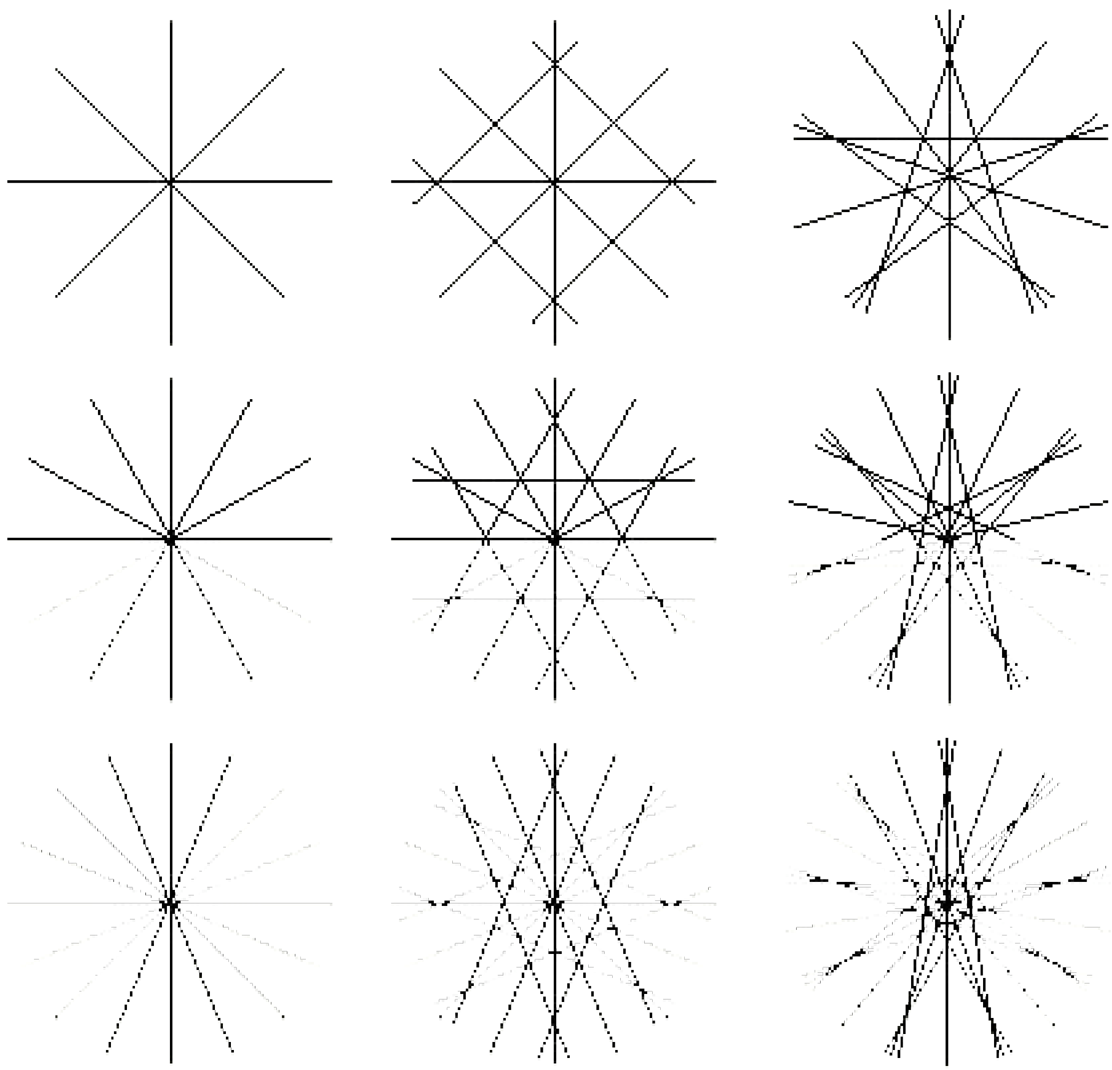


Figure 13.

Conjecture 11. There exist precisely 90 sporadic simplicial arrangements in the real projective plane.

These 90 types are described in G-1971, with two corrections needed: First, arrangement $A_7(16)$ should be deleted, since it is combinatorially equivalent to the arrangement $A_2(17)$. Second, an arrangement $A(16)$ should be added; it is shown in Figure 2.3 of G-1972. These corrections are listed in GS-84; several examples of sporadic simplicial arrangements are illustrated in Figure 1 of GS-1984. The most complicated sporadic simplicial arrangement known has 37 lines; it is shown in Figure 15.

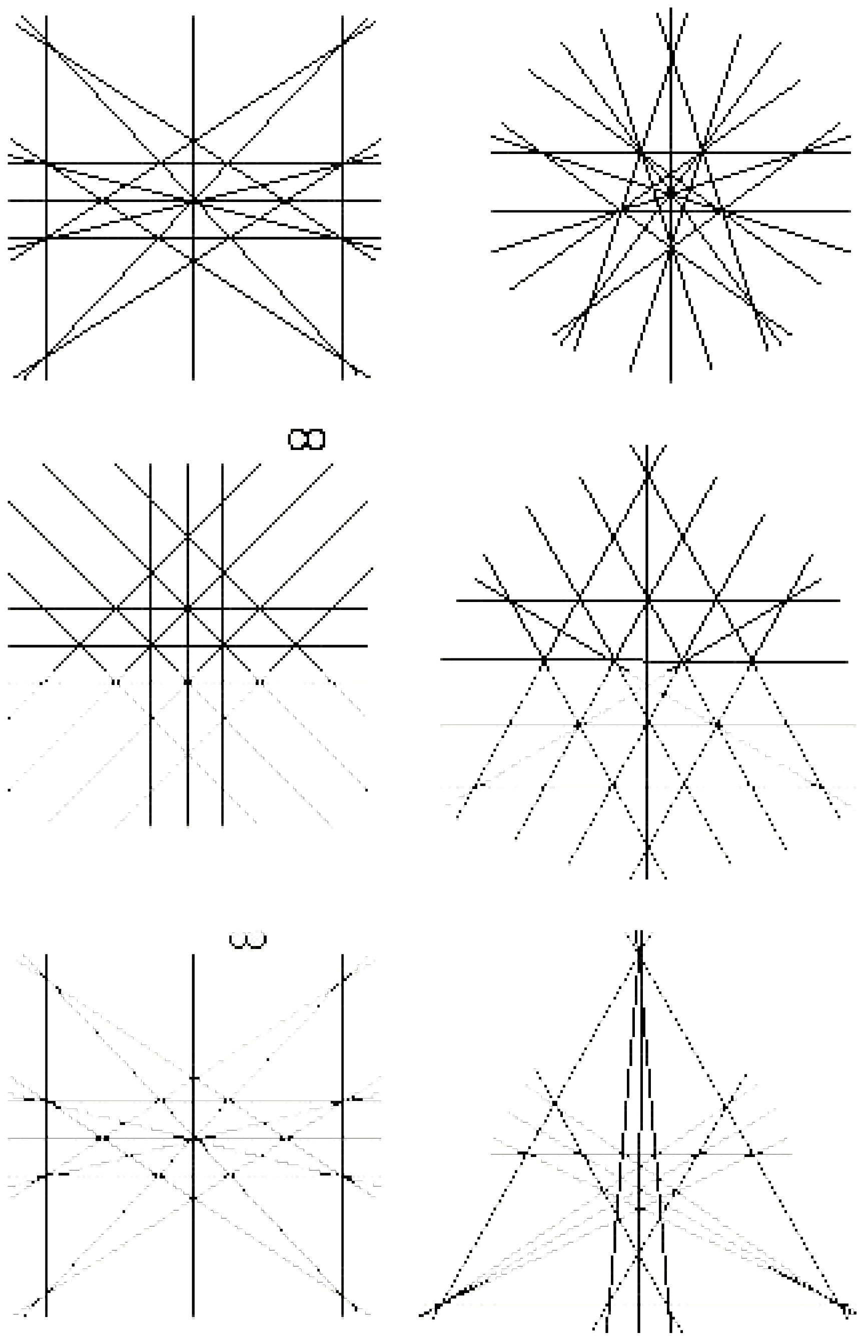
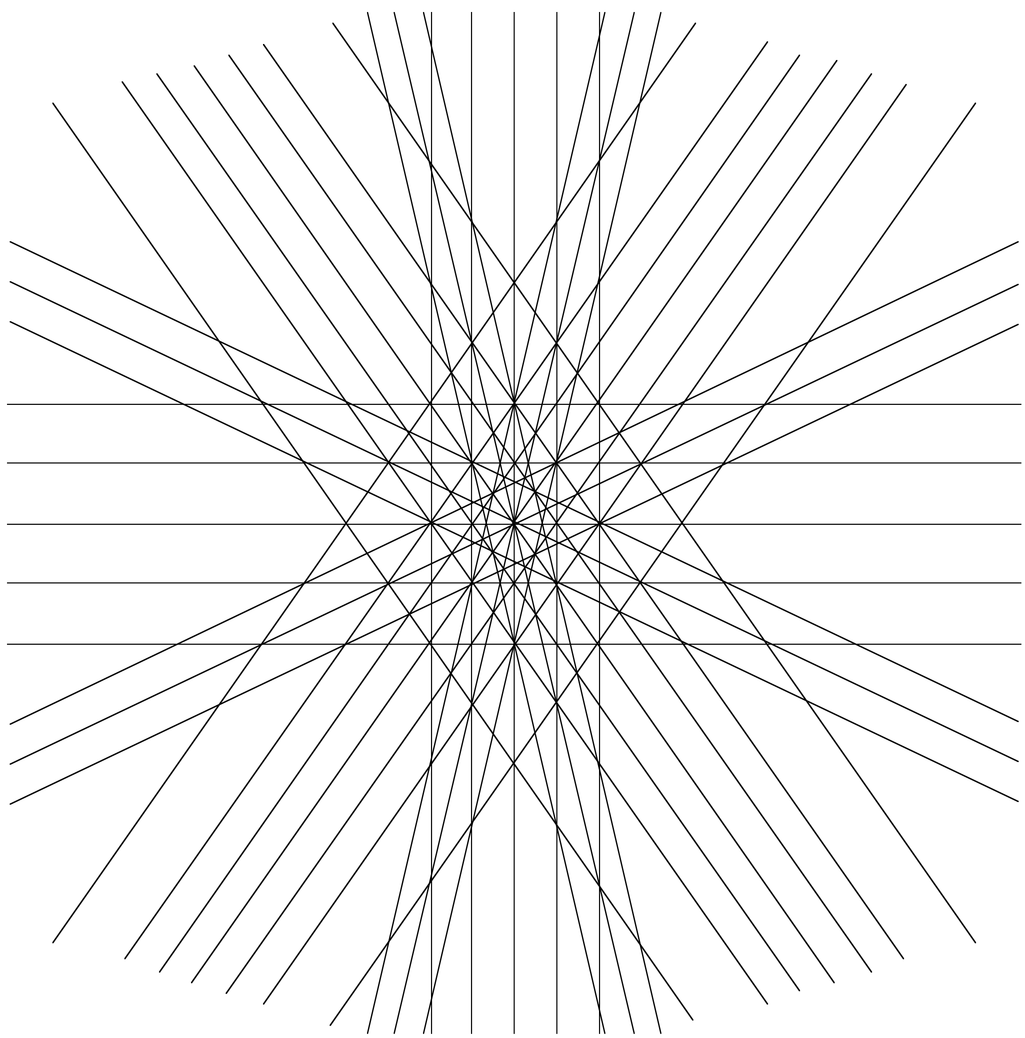


Figure 14.

While it is easily seen that there is a great profusion of simplicial arrangements of pseudolines, it is not clear whether the number of infinite families of such arrangements is finite, nor whether the number of sporadic ones is finite. (Obviously, a precise definition of "infinite family" is needed here -- but this is not what holds the solution out of reach.) A few examples are shown in Figure 16.

An arrangement of (d-1)-dimensional hyperplanes of the projective d-space is the **indecomposable** if its its hyperplanes do not arise as the joins of the (j-1)-dimensional hyperplanes in a j-dimensional subspace with the (k-1)-dimensional hyperplanes of a k-dimenional subspace skew to the former, where j + k = d-1. Clearly, near-pencils (for d = 2) are decomposable; they correspond to j = 0, k = 1. The join of a j-dimensional simplicial arrangement with a k-dimensional simplicial arrangement yields, as is easily seen, a (j+k+1)-dimensional simplicial arrangement. Concerning simplicial arrangements in higher-dimensional projective spaces we have:



Line at infinity included

Figure 15.

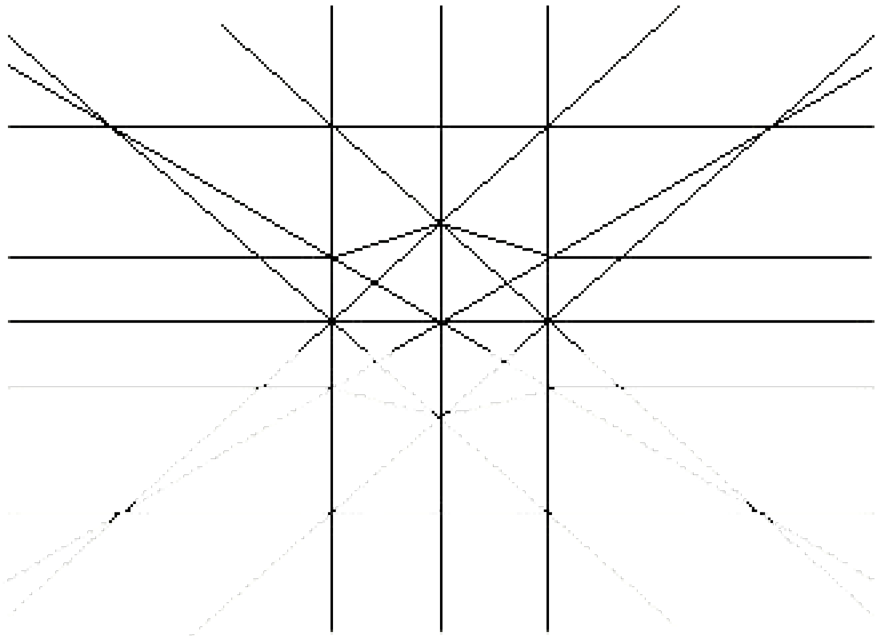


Figure 16.

Conjecture 12. For every $d \geq 3$ there is only a finite number of indecomposable simplicial arrangements in the projective d-space.

For other results and conjecture about higher-dimensional simplicial arrangements, and for references to other related works, see GS-1984.

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8. Venn diagrams are special families of curves, which lead to many interesting questions. I have been working on same of these for a long time, and I am happy to report that there have been several significant developments in recent years; see Ruskey [1997] for details. Here is a brief account.

Given a family $\mathbb{C} = \{C_1, C_2, ..., C_n\}$ of n simple (Jordan) curves which intersect pairwise transversely in finitely many points, we say that it is an **independent family** if each of the 2^n sets

$$X_1 \cap X_2 \cap ... \cap X_n \qquad (*)$$

is non-empty, where X_j denotes one of the two connected components of the complement of C_j (that is, each X_j is either the interior or the exterior of C_j). An independent family $\mathbb{C}$ is a **Venn diagram** if each of the sets in (*) is connected. An independent family or Venn diagram is called **simple** if no three curves have a common point.

A Venn diagram with n curves is said to be **symmetric** if rotations through $360/n$ degrees map the family of curves onto itself, so that the diagram is not changed by the rotation. This concept was introduced by Henderson [1963], who provided two examples of non-simple symmetric Venn diagrams for $n = 5$. A simple symmetric Venn diagram consisting of five ellipses was given in G-1975; it is shown in Figure 17. As noted by Henderson, symmetric Venn diagrams with n curves cannot exist for values of n that are composite. Hence $n = 7$ is the next value for which a symmetric Venn diagram might exist. Henderson stated that such a diagram has been found; however, at later inquiry he could not locate it; I could not find one either, and in G-1975 I conjectured that such diagrams do not exist.

As it turned out, this conjecture was definitely wrong. In G-1992 I presented two combinatorially distinct simple symmetric Venn diagrams with $n = 7$. (In one of the curious coincidences, soon thereafter several other people found various other counterexamples for $n = 7$; Ruskey [1997] for details and references.) The examples I found made me change my mind regarding the existence of symmetric diagrams, and formulated the new conjecture:

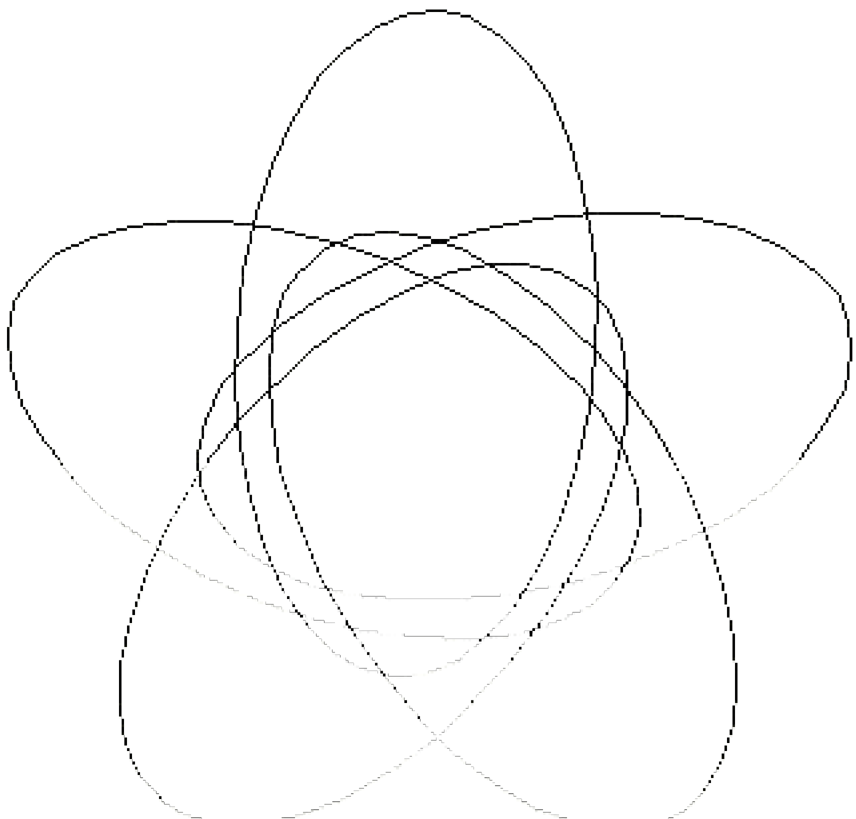


Figure 17.

Conjecture 13. For every prime n there exists a symmetric Venn diagram with n curves.

In fact, I would like to strengthen this formulation to assert the existence of a *simple* diagram of this kind.

Very recently, in unpublished work Peter Hamburger seems to have established Conjecture 13. However, the strengthened formulation, dealing with simple diagrams, appears to be still unsolved.

Henderson's argument that symmetric Venn diagrams cannot exist if the number of curves is a composite integer is based on the following fact from number theory: If $n = r s$, where r and s are integers greater than 1 and r is a prime number, then the binomial coefficient $\binom{n}{r}$ is not divisible by n. On the other hand, as noted in G-1999, this obstacle disappears if instead of Venn diagrams one considers independent families of n sets — however, such families seems to be of little interest since it is very easy to generate them for every n. But while it may seem, on number-theoretical or combinatorial grounds, that such families must have a very large number of regions, a closer investigation shows that as far as combinatorics and number theory are concerned, the number of regions could be not too much larger than in a Venn diagram. This happens because many of the types of regions occur in n-tuples, and only few require duplication in order to accommodate rotational invariance.

Let us denote by (a,b,...,f) a selection of the elements a, b, ... , f, from the family of labels of the members of the independent family of curves. All selections that can be transformed into each other by cyclic permutations of the labels are said to constitute a **type** of selections. Clearly, in a symmetric independent family of n curves, each type (except the selections of none, or of all labels) must be represented by n or a multiple of n regions. A discussion of the case $n = 6$ may illustrate this contention. The 12 relevant selections here are (a), (a,a+1), (a,a+2), (a,a+3), (a,a+1,a+2), (a,a+1,a+3), (a,a+1,a+4), (a,a+2,a+4), (a,a+1,a+2,a+3), (a,a+1,a+2,a+4), (a,a+1, a+3, a+4), (a,a+1,a+2,a+3,a+4). Hence there must be at least $12 \cdot 6 + 2 = 74$ regions in any symmetric independent family of six curves, instead of the 64 regions in a Venn diagram of 6 curves.

The above example can be generalized to obtain a lower bound on the number of regions that must be present in any symmetric independent family of n curves. The resulting lower bound is $M(n) = 2 + n \cdot (C_n - 2)$, where C_n is the number of distinct 2-colored necklaces of n beads, provided rotationally equivalent necklaces are not distinguished. The numbers C_n have been studied by several authors (see Sloane and Plouffe [1995], sequence M0564, where additional references can be found). From explicit formulas for the numbers C_n it it may be shown that the rate of growth of $M(n)$ is about 2^n for all n, and that if n is prime then $M(n) = 2^n$. Thus the following is a generalization of Conjecture 10, as formulated in G-1999:

Conjecture 14. For every integer n there exists a symmetric independent family of n curves with only $M(n)$ regions.

In Figure 15 we show examples of such minimal symmetric independent families of 4 and 6 curves; we note that, as is easily verified, $M(4) = 18$. and, as mentioned earluer, $M(6) = 74$. The first undecided case of the conjecture is $n = 8$.

A curious property of the known examples of minimal symmetric independent families for composite n is that none is simple. While for $n = 4$ it can be shown that no such family can be simple, it is not clear whether the same is true for $n = 6$ or higher values of n. This leads me to:

Conjecture 15. If n is not a prime, no symmetric independent family with $M(n)$ regions is simple.

* * * * *

9. A (geometric) **(n_k) configuration** is a family of n points and n (straight) lines in the Euclidean plane such that each point is on precisely k of the lines, and each line contains precisely k of the points. While the study of geometric and combinatorial (n_3) configurations goes back more than a century, very little has been

written about geometric (n_k) configurations for $k \geq 4$. It is well known that there exist (n_k) configurations for each k and for some suitably large n. However, these arguments yield only configurations with very large values of n, and their size makes it quite pointless to represent them in the plane. Recently, G-2000, I reviewed the information available to me concerning the question: *For which* n *do there exist* (n_4) *configurations ?*

To formulate answers to this question, it is appropriate to introduce an additional concept. An (n_k) configuration is said to be **connected** if it is possible to

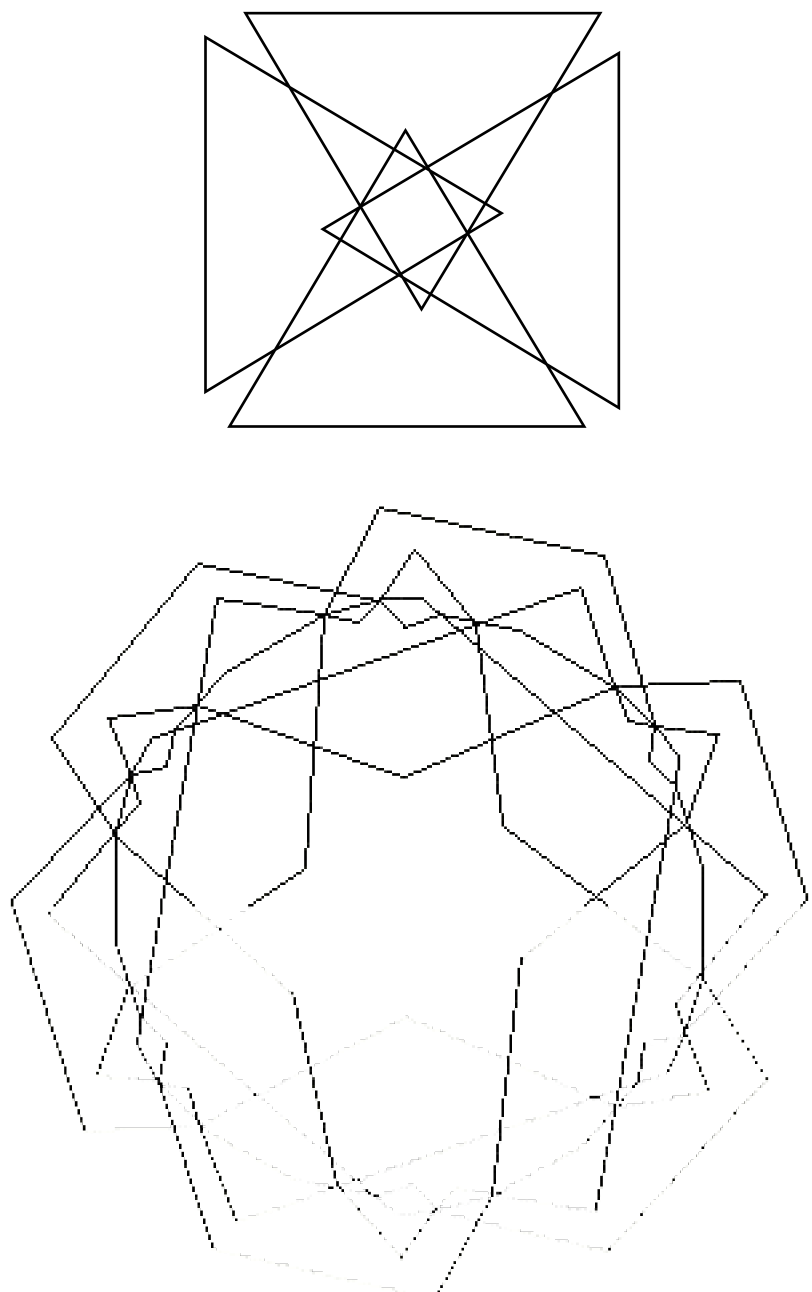


Figure 18.

reach every point starting from an arbitrary point and stepping to other points only if they are on one of the lines of the configuration. Equivalently, it is connected if it is not the union of two configurations with the same k but smaller n. The six configurations shown in Figure 19 are examples of connected (36_4) configurations while the (48_4) configuration shown in Figure 20 is not connected.

Combining several methods of construction of connected (n_4) configurations I found (see G-2000a) that such configurations are quite plentiful. Without explicitly saying so in G-2000a, I was convinced in the validity of:

Conjecture 16. Connected (n_4) configurations exist if and only if $n \geq 21$ and is none of the following: $n = 32$ or $n = p$ or $n = 2p$ or $n = p^2$ or $n = 2p^2$ or $n = p_1p_2$, where p, p_1, p_2 are odd primes and $p_1 < p_2 < 2p_1$.

However, even before G-2000a appeared in print I discovered a new family of constructions. Using these I can show that for a sufficiently large n^*, there are connected (n_4) configurations for every $n \geq n^*$. In fact, it is enough to take $n^* = 17,500$. It appears likely that a much smaller value of n^* is sufficient, and I believe that there are fewer than 100 values of n for which no connected (n_4) configurations exist. In particular, the following are the only 37 values of $n \leq 100$ for which the existence of such a configuration is undecided: 15, 16, 17, 18, 19, 20, 22, 23, 25, 26, 29, 31, 32, 34, 37, 38, 41, 43, 46, 47, 49, 53, 58, 59, 61, 62, 67, 71, 77, 79, 82, 83, 86, 89, 94, 97, 98.

It should be noted that it is known that no geometric (n_4) configuration exists if $n \leq 14$, and that (connected) *combinatorial* (n_4) configurations exist for every $n \geq 13$. If disconnected configurations are admitted, then the list of values of $n > 21$ for which no (n_4) configuration is known becomes very short:

Conjecture 17. There exists no geometric (n_4) configuration for any of the following fifteen values $n = 22, 23, 25, 26, 29, 31, 32, 34, 37, 38, 41, 43, 46, 47, 53$.

It is known that geometric (n_4) configurations exist for all other $n > 21$.

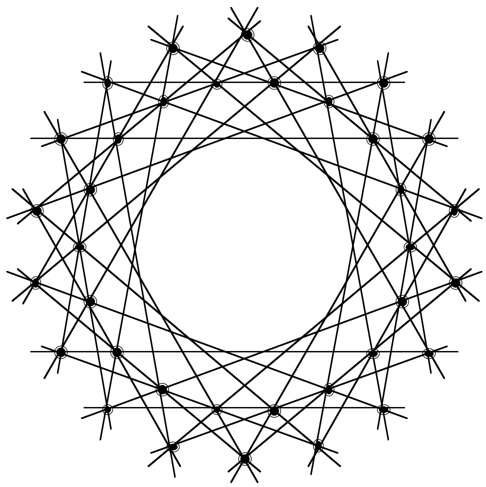
The situation concerning (n_k) configurations for $k \geq 5$ seems much less clear; the literature contains almost no information. I believe that there is a radical difference between $k = 5$ and $k > 5$, and I propose:

Conjecture 18. There exist connected (n_5) configurations for all but a finite number of values of n. For each $k \geq 6$, there are infinitely many values of n for which no connected (n_k) configuration exists.

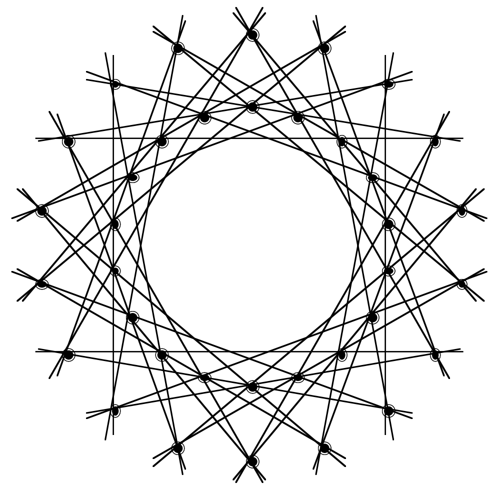
* * * * *

10. I'll conclude by discussing another topic in which shooting from the hip led me to wrong guesses -- but led to very interesting developments and holds promise for much more. **Rhombic isohedra** are polyhedra having as faces congruent rhombi, all equivalent under isometric symmetries of the polyhedron. Convex rhombic isohedra are well known, and so are the three rhombic **triacontahedra** -- one convex and two selfintersecting in the manner of the Kepler-Poinsot regular polyhedra (see Coxeter [1973], page 101). A remarkable acoptic rhombic **hexecontahedron** was first described by Unkelbach [1940]; it is shown in Figure 21.

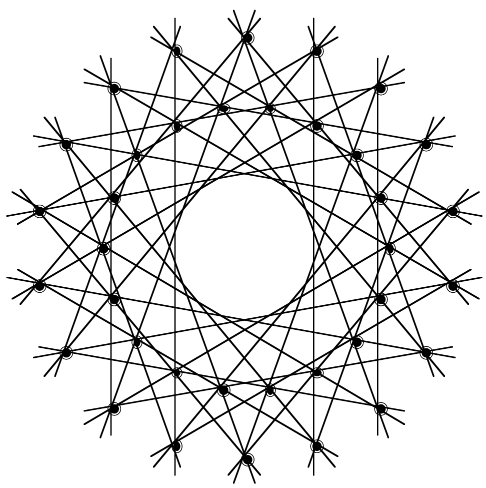
In G-1996a,b I described a rhombic hexecontahedron (shown in Figure 22) which is analogous to the two nonconvex triacontahedra in that it is selfintersecting in the manner of the Kepler-Poinsot polyhedra. (Following some other authors on related matters, in my ignorance of Greek I adopted the spelling I found and used "hex**e**contahedron". This earned me several letters of rebuke from people who know



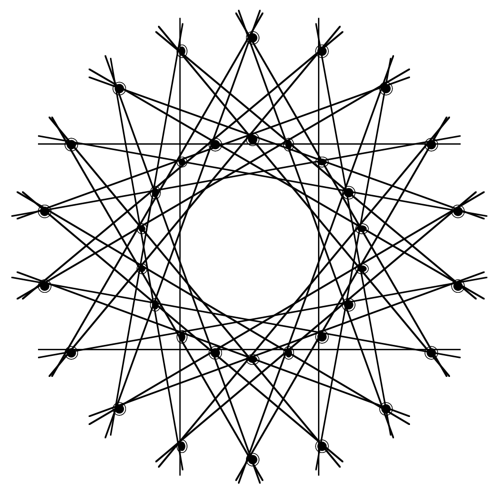
$(36_4)_1$ $18\#4_1; 6_5$



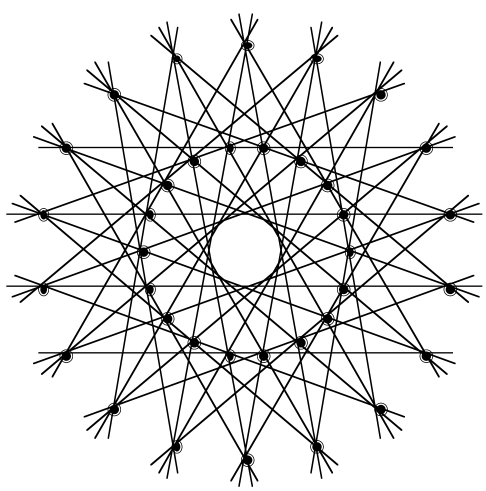
$(36_4)_2$ $18\#5_1; 6_4$



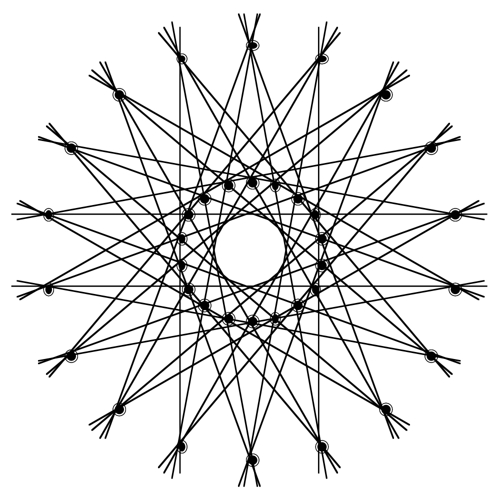
$(36_4)_3$ $18\#5_2; 7_6$



$(36_4)_4$ $18\#6_2; 7_5$



$(36_4)_5$ $18\#6_1; 8_7$



$(36_4)_6$ $18\#7_1; 8_6$

Figure 19.

better!) But worse than the spelling misstep, after failing to find another rhombic hexecontahedron, I made the following conjecture:

Conjecture 19. There exist no rhombic isohedra with 60 faces other than the two in Figures 21 and 22.

However, very soon I came to realize that this conjecture is not just wrong, but grievously wrong. This may sound somewhat oxymoronic — but the truth is that there is not just one counterexample, but quite a few — close to a dozen. It is hard to pin down the precise number, since it depends on somewhat arcane details of the definition of "polyhedron". Two of these are selfintersecting in the same way as the first one I found; they were described in G-1997, and are shown in Figure 23. Although terrible from the point of view of the conjecture, this development yielded two benefits.

First, it led to a systematic way, using Möbius nets, of generating various rhombic isohedra, as well as other isohedral polyhedra. This method has been described in G-2000b, where it is used to investigate parallelogram-faced isohedra with octahedral symmetry. The same approach can be used for the icosahedral case -- but questions relevant to the second point, to be discussed below, need to be addressed first. Möbius nets have been used extensively for more than a century in the investigation of isogonal polyhedra; this makes it very hard to understand why did their applicability to the study of isohedral polyhedra escape attention. Following G-2000b, the method has been used by Coxeter-Grünbaum [2000b] to investigate rectangle-faced hexecontahedra, by Grünbaum-Shephard [2000] for dart-faced isohedra, and by Shephard [2000] for isohedra with equilateral triangles as faces.

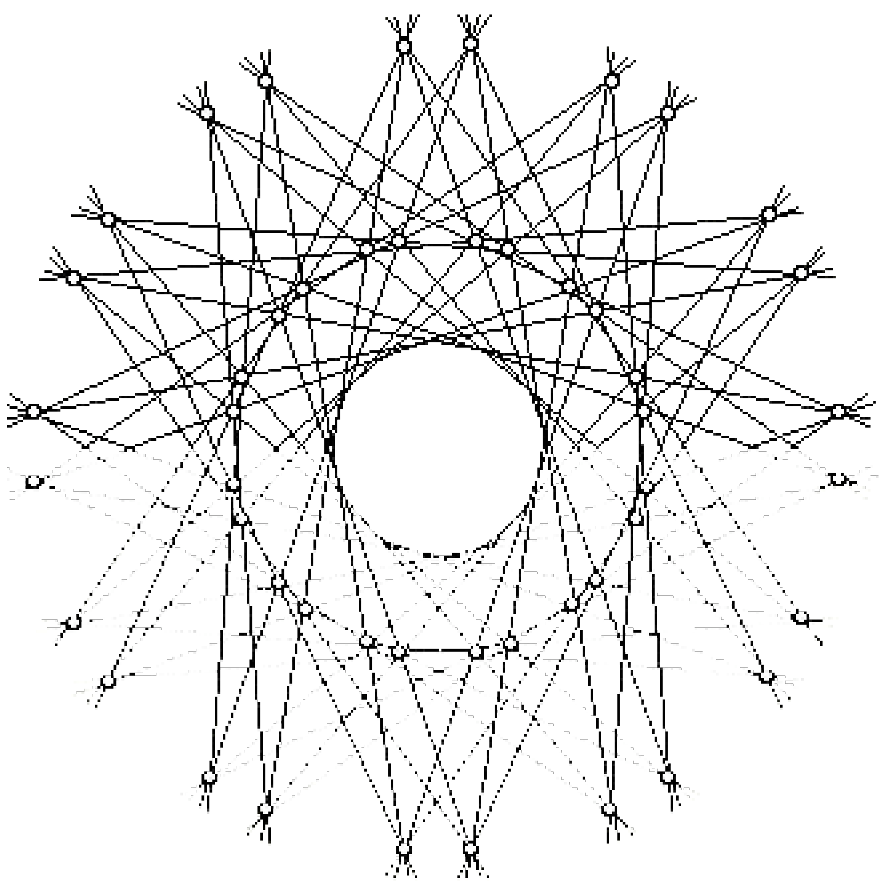


Figure 20.

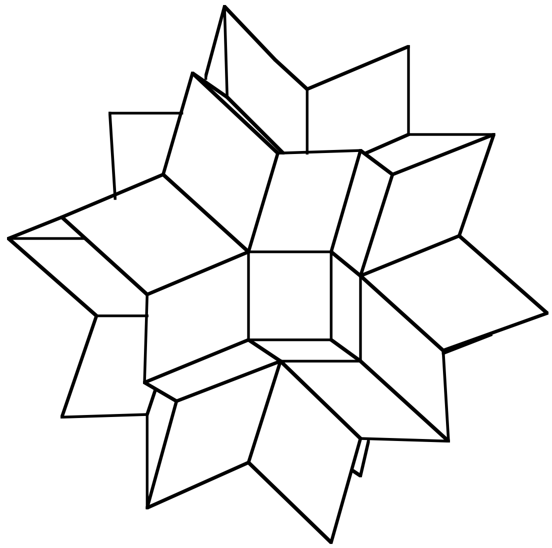


Figure 21.

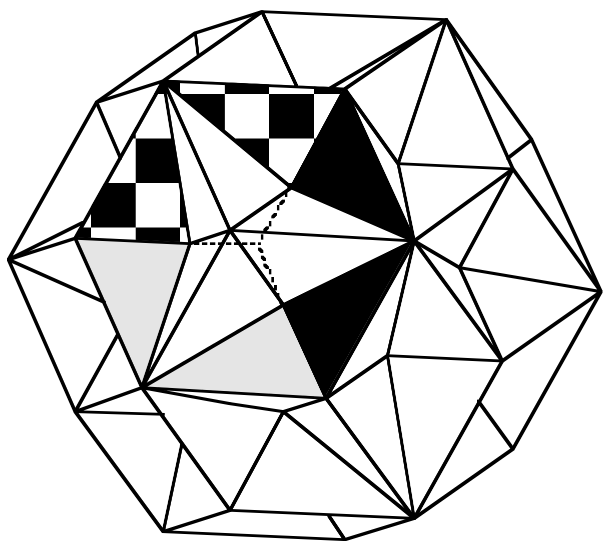


Figure 22.

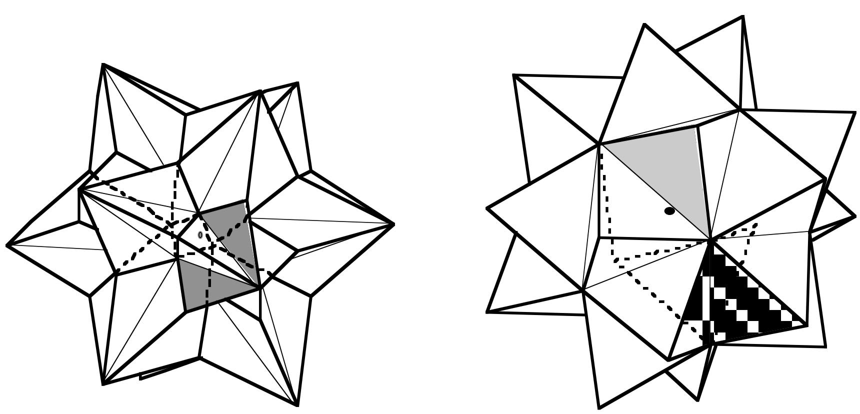


Figure 23

Second, the investigations showed the need to give precise definitions in specifying various classes of polyhedra more general than the ones usually considered. In several other investigations, the same need became apparent -- see, for example, G-1998b, Coxeter-Grünbaum [2000a] In particular, it turned out that that there is in all literature no usable definition of general geometric polyhedra other than ones with very high degree of symmetry -- such as the Kepler-Poinsot polyhedra, or the isogonal polyhedra. The search for acceptable and sensible definitions turned out to be more complicated than one would assume ahead of time. However, it also leads to discoveries of many interesting kinds of polyhedra. I hope that it will catch the interest of others.

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Branko Gr”unbaum

My favorite conjectures

Abstract

Explicitly formulated conjectures often provoke reactions and so lead to i
ncreased knowledge -- regardless of whether they are confirmed or dispro
ved. Over the years I have made many conjectures. They experienced va
ried fates: some were proved, others demolished, and still others ignored.
I will discuss several of these conjectures, the ones closest to my heart,
that deal with polyhedra, configurations, tilings and other topics of combi
natorial geometry. For each, I shall describe the present status, and in so
me cases suggest additional conjectures.

Two instances where I proved myself wrong:

Grünbaum code versus Ruskey code

2. Rhombic hexecontahedra

Rigidity of frameworks

Nonexistence of astral n_6 configs.

Heesch number is 3