**ARE ALL CONVEX POLYHEDRA TETRAGONALIZABLE?**

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A convex polyhedron P is *tetragonalizable* by a family of pyramids {F}j, each of which shares its base with a face of P, provided that for each edge E of P the two faces of the pyramids that are incident with E are coplanar and form a convex quadrangle. The convex hull Q of the union of all these pyramids is said to be a *tetragonalization* of P; as sets, it obviously contains P. The edges E of P are not edges of Q; they form one diagonal in each face of Q. The other diagonals of the faces of Q determine a graph such that it defines a polyhedron P\* dual to P.

Some examples of tetragonalizations are easily verified:

 The cube is a tetragonalization of a regular tetrahedron, see Figure 1.

 The rhombic dodecahedron is a tetragonalization of the cube, see Figure 2; it is also a tetragonalization of the regular octahedron.

  

 Figure 1. Figure 2.

 The rhombic triacontahedron is a tetragonalization of the regular icosahedron, as shown in Figure 3, or of the regular dodecahedron.

 

 Figure 3

In view of these and other examples it is natural to ask whether every convex polyhedron admits a tetragonalization. The answer to this question is not known, but we make the following guess:

**Conjecture 1**. Every convex polyhedron is tetragonalizable.

No counterexample to Conjecture 1 is known at present. On the other hand, some partial results support the conjecture. From the definition it is immediate that if P1 and P2 are affinely equivalent by an affinity , then either both are tetragonalizable (and their tetragonalizations are affinely related by ), or neither is tetragonalizable. This implies, in particular, that all tetrahedra are tetragonalizable.

The following two results support Conjecture 1:

**Theorem 1**. Every simplicial convex polyhedron admits a tetragonalization.

(A convex polyhedron is *simplicial* if all its faces are triangles.)

Proof of Theorem 1. For a convex polyhedron P let Pe be the external bisector of the dihedral angle determined by the pairs of faces incident with an edge e of P. The collection of all such planes determines a convex polyhedron Q. The three planes determined by the edges of a face F meet at a point VF that is above the face F; each such point is a vertex of Q. For each edge e its two vertices and the two points VF and VF\* determine a convex quadrangle Te that is a face of Q, and that contains the edge e as one of its diagonals; the other diagonal of Te connects the vertices VF and VF\*. The polyhedron Q is a tetragonalization of P since Te is the union of the triangles spanned by e and the two vertices VF and VF\*. These triangles are sides of the pyramids on the faces F and F\*, with vertices VF and VF\*, respectively. This completes the proof of Theorem 1. ◊

**Theorem 2**. Every midscribable convex polyhedron P admits a tetragonalization.

The proof of theorem 2 requires some preliminary definitions and facts. Let P be a convex polyhedron. We say that P is *inscribable (midscribable, or circumscribable)* provided there exists a sphere S that contains all the vertices of P (for which all edges of P are tangent to S, or is such that all facets of P are tangent to S, respectively). In such situations we shall also say that P admits a circumsphere, midsphere, or insphere, respectively. It is clear that every regular polyhedron has all these properties, and that the three spheres are concentric.

Remarkably, this result has a converse, which can be proved by straightforward arguments (given in detail by Brückner [1, p. 123], Coxeter [2, p. 157]): If a convex polyhedron P admits an insphere, as well as a midsphere and a circumsphere, and if these three spheres are concentric, then P is one of the five regular polyhedra.

It is worth mentioning that there are polyhedra that are not regular but for which all three spheres exist. However, for them the spheres are not all concentric. The simplest examples of this kind are the right pyramids.

**Conjecture 2.** Pyramids and the regular polyhedra are the only convex polyhedra that admit all three spheres.

A polyhedron P is said to be of **inscribable** (**midscribable**, or **circumscribable**) **type** provided it is combinatorially equivalent to a polyhedron that is inscribable, or midscribable, or circumscribable, respectively.

It appears to have been frequently assumed that certain general classes of polyhedra are of inscribable or circumscribable type. Jacob Steiner posed the question whether every polyhedron is of circumscribable type. In this context, Brückner [1, p. 163] claimed that every simplicial polyhedron is of inscribable type. However, Steinitz [7] showed that not only is the answer to Steiner’s question negative, but that Brückner’s assertion is invalid as well. From the exceedingly elegant –– though completely elementary –– results of Steinitz if follows that, for example, the triakis octahedron [3.8.8] (which is one of the Catalan polyhedra) is a simplicial polyhedron of non-inscribable type. An accessible account of the results of Steinitz [7] and their proofs is given in Grünbaum [4, pp. 285 – 286]. It may be noted that a polyhedron is of inscribable type if and only if its duals are of a circumscribable type. Hence the above shows the existence of simple polyhedra that are not of the circumscribable type. It is also easy to provethat there are polyhedra that are simultaneously of non-inscribable and non-circumscribable types.

In a private communication, the late Theodore S. Motzkin asked whether every polyhedron is isomorphic to one in which every facet has circumcircles. Again, the answer is negative even for simple polyhedra such as the one in Figure 4; this can be used as an alternative approach to Steinitz’s results mentioned above. (See Grünbaum [3], or [4, p. 287].)



Figure 4.

All the previous facts were presented mainly to serve as a background to the remarkably different situation concerning midspheres. The unexpected result here is that not only is every polyhedron of midscribable type, but that the following stronger result holds:

**Lemma.** If P is a convex polyhedron, there exists a *unique* polyhedron Q of the same combinatorial type as P, that has all the following properties:

1. Q has a midsphere S;
2. The barycenter (center of gravity) of the points of tangency of the edges is the center of the midsphere;
3. S is the midsphere of the polar Q^ of Q with respect to S, the tangency points of the edges of Q^ coincide with those of Q, and the corresponding edges are perpendicular to each other.

Clearly, Q depends only on the combinatorial type of P and not on the particular polyhedron of that type. The uniqueness of Q has to be understood as being up to position and size.

As an illustration of the above we show in Figure 5 a pair of polar polyhedra, and in Figure 6 the correspondind polyhedra Q described in the Lemma.

The various published proofs of the Lemma rely mostly on deep results concerning packings of circles in the plane or on the sphere. Such packings have been considered long ago in connection with functions of a complex variable (starting with Koebe in 1936), and have been greatly advanced through the work of Andreev, Thurston, Colin de Verdiére, Breitwell and Scheinerman, Mohar,



Figure 5.



 Figure 6.

Bobenko and Springborn, and Ziegler, among others. Detailed references, with accounts of the results of the various researchers, and with proofs, can be found in Mohar and Thomassen [5,Section 2.8] and Ziegler [8, Lecture 4]. All the proofs are too complex to be included here.

As an immediate consequence of the Lemma we have:

For every polyhedron there is an isomorphic one all facets of which have incircles. This is in contrast to the answer to Motzkin’s problem about circumcircles, discussed above. Thus midspheres behave quite differently from inspheres and circumspheres. This becomes even more remarkable when contrasted with the fact, proved by Schulte [6], that for convex d-polytopes of dimension d ≥ 4, faces of any dimension might fail to be of the appropriate tangential type.

We return now to the proof of Theorem 2. Given P, we claim that the polyhedron Q described in Lemma 2 is tetragonalizable. Indeed, consider the sphere S of Q. Any face F of Q is circumscribed about the circle C that is the intersection of S with the plane of F. The points of contact of C and F are the points of tangency of the edges E of F to the sphere S. Therefore the planes tangent to S at all such points of F meet at a unique point VF. Taking the convex hull of F and VF yields a polyhedron such that for pyramids over faces that share E the sides that contain E are coplanar. Thus, Q admits a tetragonalization. ◊

Clearly, this implies the validity of the claim that every convex polyhedron P is combinatorially equivalent to a convex polyhedron R that admits a tetragonalization.

We note that if Q is a tetragonalization of a convex polyhedron P, then vertices of Q that are not vertices of P correspond to the faces of P, and the diagonals of the faces of Q that do not coincide with edges of P determine circuits that correspond to the vertices of P. Hence the convex hull P\* of the vertices of Q that are not vertices of P is a dual of P. It follows that Q is a tetragonalization of P\* as well. These observations are illustrated by Figures 1, 2, and 3.

Another natural question concerns the uniqueness of tetragonalization of a given polyhedron, or lack of it. For the

regular tetrahedron, octahedron and cube it is easy to show that there are uncountably many distinct tetragonalizations. On the other hand, no example of a polyhedron is known that admits only a unique tetragonalization. Hence we venture:

**Conjecture 3**. Every tetragonalizable convex polyhedron admits a continuum of tetragonalization.

As a partial confirmation of Conjecture 3 we have:

**Theorem 3**. Cubes admit continuum many tetragonalizations.

Proof. Assuming the cube has vertices (±1,±1,±1) with all choices of signs, the six points (2,0,t), (-2,0,t), (0,0,2-t), (0,0,-2-t), (0,2,t), (0,-2,t) are the apices of the pyramids that yield a tetragonalization of the cube for every value of t in the interval –1 < t < 1. Similar (but more complicated) calculations show that every point (x,y,t) with –2 < x + y < 2, –1 < t < 1 can play the role of (2,0,t) in the above construction. ◊

For regular tetrahedra and octahedra, as well as for dodecahedra and icosahedra the situation is similar although algebraically more involved. Regular pyramids and prisms, as well as many other polyhedra, can be shown to have continuum many tetragonalizations.

Due to the absence of counterexamples, we also make:

**Conjecture 4**. Every quadrangle-faced convex polyhedron is a tetragonalization of some convex polyhedron.

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