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QUADRANGLES, PENTAGONS, AND COMPUTERS, REVISITED

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This is an update of the note [2]. To make the present remarks selfcontained, I'll briefly recall the topic and the problems from [2].

The perpendicular bisectors of the sides of a quadrilateral Q form in general another quadrilateral Q_1, the vertices of which are the intersections of the bisectors of adjacent sides of Q. We shall call this mapping of quadrangles to quadrangles the *bisector construction* and denote it by β; thus $Q_1 = \beta(Q)$. In 1953 J. Langr [3] asked:

(i) Show that $Q_2 = \beta(Q_1) = \beta(\beta(Q))$ is similar to Q; and

(ii) Find the ratio of similarity of Q_2 and Q.

At the time [2] was written the only solutions of (i) I was aware of were obtained by using computers in various ways; see [2] for details and references (as well as for the meaning of the words "in general" used above). Since then several people have informed me that they have found "traditional" proofs of (i), and some related results. However, the most interesting new development is the solution of (ii) by G. C. Shephard [7]. He not only gives an answer to (ii), but also provides other information that was not noticed previously. Nevertheless, it is a strange answer, and quite mysterious as to its geometric meaning.

In order to describe Shephard's results we need the concept of *deflection* at a vertex of an oriented quadrangle $Q = [V_1V_2V_3V_4]$. We start by extending the side $V_{i-1}V_i$ beyond V_i, to W_i. The deflection θ_i of Q at V_i is the angle, less that π, through which the ray V_iW_i has to be turned in order to coincide with the ray V_iV_{i+1}; the deflection is counted as positive if the turn is counterclockwise, and negative otherwise. For example, all deflections are positive in Figure 1(a), while

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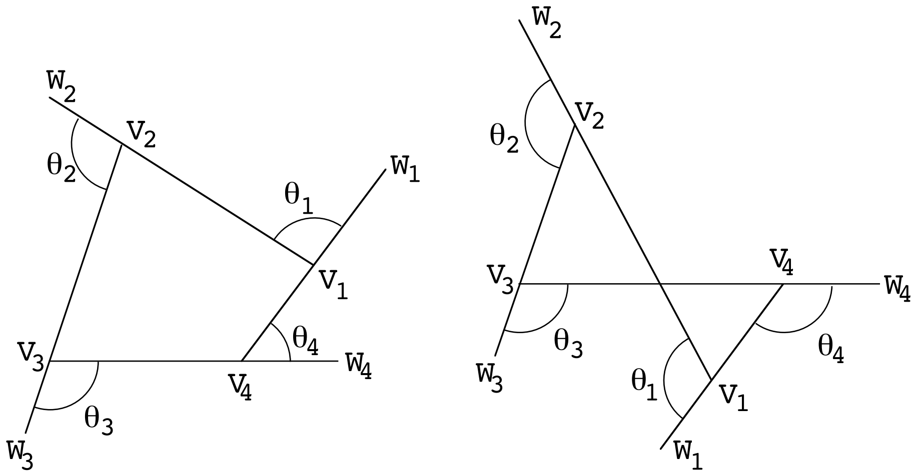


(a) (b)

Figure 1.

in Figure 1(b) only θ_2 and θ_3 are positive, the other two are negative. Clearly, $\Sigma_i\ \theta_i + 0 \pmod{2\pi}$ in all cases. Shephard's result is the determination of a number we shall denote by μ, the absolute value of which is the similarity ratio of Q_2 to Q; clearly, this ratio is unchanged by cyclic permutations of of the vertices of Q, or reversal of orientation. Shephard's result can be formulated as:

(a) $|\mu|$ depends only on the deflections θ_i.

(b) The value of μ is given by

$$-8\mu = \Sigma_j \frac{1}{\sin^2\theta_j} + \frac{\sin\theta_1 \sin\theta_3 + \sin\theta_2 \sin\theta_4}{\sin(\theta_1+\theta_3)\sin(\theta_1+\theta_4)} \cdot \Sigma_j\ (-1)^j \sin^2\theta_j$$

While (a) is interesting, the formula in (b) is really rather strange; any interpretation of its relation to other geometric aspects of the quadrangles in question would be most welcome.

The definitions of the mapping β and of deflections can be extended to pentagons (or polygons of any other number of sides) in the obvious way. As mentioned in [2], the unexpected property of β for pentagons P is that $P_3 = \beta(P_2) = (\beta(\beta(\beta(P)))$ is similar to P_1. So far there has been no proof of this fact by traditional methods; moreover, here we are still lacking the analog of Shephard's evaluation (b) of the ratio of similarity. On the other hand, computational evidence is that Shephard's result (a) does extend to pentagons. There has been no advance in our understanding the relationship between the shapes of pentagons and of their descendants under the β transformation, or of the situation concerning the application of β to polygons with six or more sides.

It seems appropriate to discuss here another topic. As far as I know it has no direct connection with Langr's problem, but there are so many analogies in the structure of the known facts and open problems that I feel the two topics should be juxtaposed — possibly even considered as parts of a more general investigation of transformations among polygons.

Starting from a convex pentagon $P = V_1V_2V_3V_4V_5$, by drawing its diagonals another convex pentagon $P_1 = D_1D_2D_3D_4D_5 = \delta(P)$ is obtained (see Figure 2); we may call this the *diagonal construction*. I. J. Schoenberg (in [5, p.102] and in some earlier publications) conjectured that the sequence of pentagons P, $P_1 = \delta(P)$, $P_2 = \delta(\delta(P))$, $P_3 = \delta(\delta(\delta(P)))$, ... always converges to a single point, say $\lambda(P)$. (Since $P_1 = \delta(P)$ is contained in the interior of P, it is easily seen that this sequence converges either to a point, or to a segment.) Schoenberg's

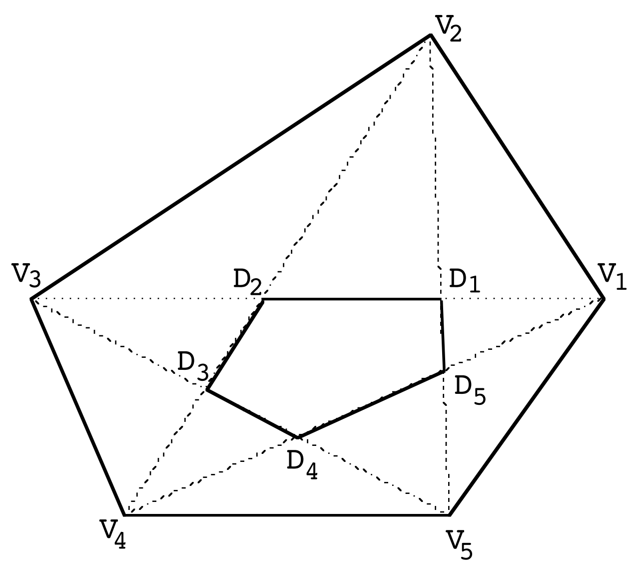


Figure 2.

conjecture was established independently by Moran [4] and Schwartz [6]. Both proofs start by establishing that $P_1 = \delta(P)$ is projectively equivalent to P. By this is meant that there is a projective transformation τ of the extended plane (that is, the projective plane; see, for example, Coxeter [1]) which maps points to points and lines to lines, such that the action of δ on P coincides with that of the diagonal construction, $\tau(P) = \delta(P)$. Therefore the iterations of δ coincide with those of τ, from which the affirmative answer to Schoenberg's problem may be deduced. However, just as the similarity ratio in the solution of Langr's problem is quite complicated, so is the determination of the limit point $\lambda(P)$ in Schoenberg's problem. The description of $\lambda(P)$ would leads us too far afield, and the interested reader is urged to consult [6].

The diagonal construction can be applied to polygons with more that five sides. Parallel to the situation concerning the bisector construction for pentagons is the result of Schwartz [6] that $\delta(\delta(H))$ is a projective image of the polygon H^* obtained by a suitable renaming of the vertices of the convex polygon H, so that, in fact, $\delta(\delta(\delta(\delta(H))))$ is a projective image of H. This is illustrated in Figure 3. However, just as the behavior of the bisector construction on pentagons is still not fully understood, so is the behavior of the diagonal construction on hexagons. The analogy extends to polygons with more sides — in either case there are tantalizing hints of the facts, but little that is certain; in particular, there is experimental evidence that no iteration leads to a similar or projectively equivalent polygon, although some near-periodicity phenomena can be observed.

One additional property of the diagonal construction that is indicated by computational experiments is the following. Let $\rho(P)$ denote the product of the cross-ratios $\prod_i [V_i,V_{i+2};D_i,D_{i+1}]$, where V_i's are the vertices of the polygon P, and D_i's are the intersections of the diagonals, as in Figure 2. Since in the case of pentagons P and $\delta(P)$ are projective images of each other, obviously $\rho(P) = \rho(\delta(P))$. However, numerical evidence shows that that the relation $\rho(P) = \rho(\delta(P))$ holds for *all* convex polygons P, regardless of the number of sides, although for polygons P with more than five sides P and $\delta(P)$ are not projectively related. It would be very interesting to have a proof of this observation.

It should be noted that for convex polygons with 7 or more sides, the diagonal construction can be complemented by analogous constructions that use longer diagonals (see Figure 4a). These appear to

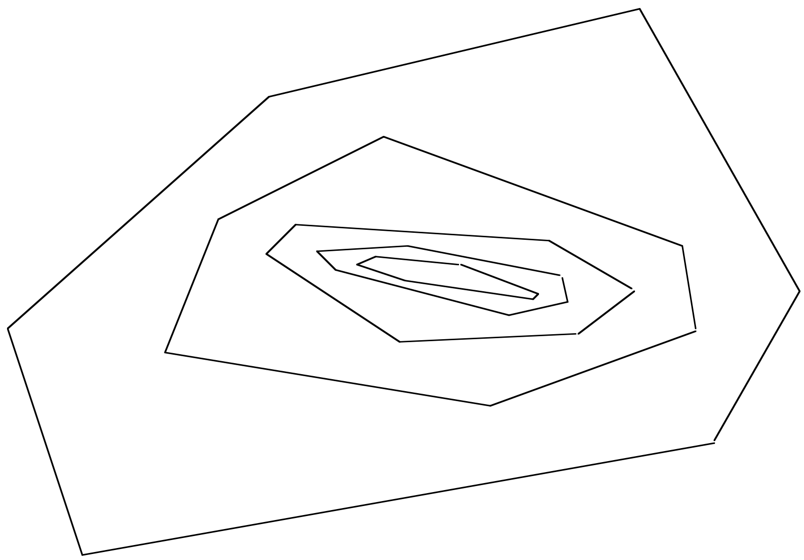
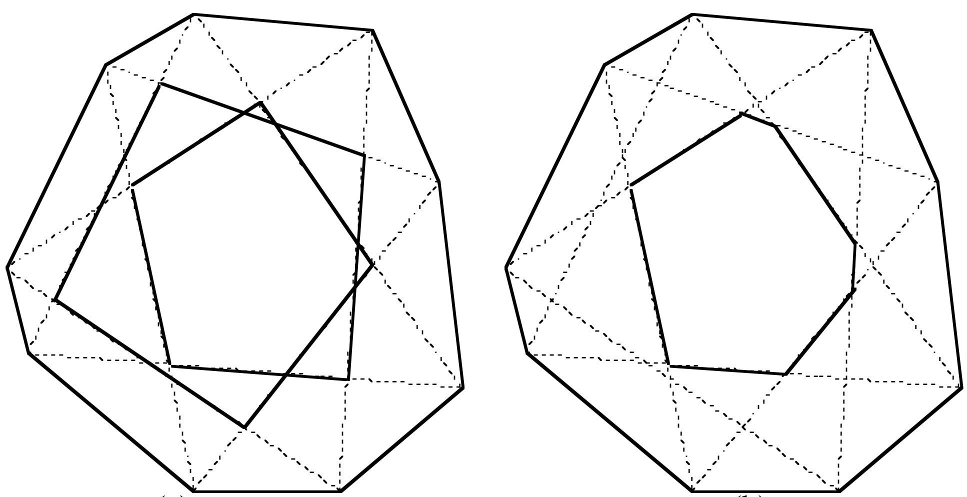


Figure 3.



(a) (b)

Figure 4.

involve nonconvex polygons even if is the starting polygon is convex. Other constructions that can be considered as analogs of the diagonal construction can lead from convex polygons to convex polygons, possibly with fewer sides than the starting one (Figure 4b). All such constructions remain completely unexplored.

To end with some general observations: I believe that the attraction geometry holds for many people stems, in roughly equal parts, from two aspects. On the one hand, there is its visual appeal and impact, which conveys information at a much more immediate level than text or formulae. On the other hand, there is the lure of the unexpected: very small changes in formulation often lead from easy exercises to unsolved problems, whose inherent difficulty is not due to the thickness of the layers of abstraction surrounding them. The topics we have discussed in the preceding pages remain challenging, although there would have been no difficulty in making the questions understandable to Euclid or Archimedes, or in explaining them to any youngster today.

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