NOTES

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Vector Analysis Proof of Erdős' Inequality for Triangles

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Abstract. We define a new concept, the "1/2-power of plane vectors," and use it to provide another proof of Erdős' inequality for triangles.

1. INTRODUCTION. Let $\triangle ABC$ be a triangle and O a point in it. Let the distances from O to vertices A, B and C be p, q and r, and let those to sides BC, CA and AB be u, v and w. Erdős' inequality for triangles then asserts that

$$p + q + r \geq 2(u + v + w)$$

with equality holding if and only if the triangle is equilateral and O is its center.

A number of authors have given proofs for this inequality using different tools. Kazarinoff's proof [**3**] uses Pappus' Theorem, and Avez's proof [**2**] needs Ptolemy's Theorem. However, a paper by Alsina and Nelsen [**1**] provides a simple, elementary proof. Most notably, Kusco [**4**] requires only trigonometric functions in achieving his short proof. We recently provided a proof of the inequality in [**5**] using "1/2-power of plane vectors," a concept that we originated.

Our examination of the equality condition in [**5**] was insufficient, and [**5**] is only available in Japanese. Therefore, we offer our proof here after revising the last part using an argument suggested by T. Kambayashi.

2. NOTATION. Let $\mathbf{a} = (a_1, a_2)$ and $\mathbf{b} = (b_1, b_2)$ represent plane vectors throughout. We will regard these as $\mathbf{a} = (a_1, a_2, 0)$ and $\mathbf{b} = (b_1, b_2, 0)$ in $\mathbb{R}^3$. For any $\mathbf{a} = (a_1, a_2, 0) = (a\cos\theta_a, a\sin\theta_a, 0)$, define

$$\mathbf{a}^{1/2} := (a^{1/2}\cos(\theta_a/2), a^{1/2}\sin(\theta_a/2), 0) \tag{1}$$

and likewise for $\mathbf{b} = (b\cos\theta_b, b\sin\theta_b, 0)$. We begin with two lemmas.

Lemma 2.1.

(a) $\mathbf{a}^{1/2} \cdot \mathbf{a}^{1/2} = a = |\mathbf{a}|$.

(b) $\mathbf{a} \times \mathbf{b} = 2(\mathbf{a}^{1/2} \cdot \mathbf{b}^{1/2})(\mathbf{a}^{1/2} \times \mathbf{b}^{1/2})$.

(c) $|\mathbf{a} - \mathbf{b}| \geq 2|\mathbf{a}^{1/2} \times \mathbf{b}^{1/2}|$, *with equality holding if and only if* $|\mathbf{a}| = |\mathbf{b}|$.

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Proof. The proof of (a) is clear. (b) Let $\theta := \theta_b - \theta_a$. Then

$$\mathbf{a} \times \mathbf{b} = (0, 0, ab \sin\theta) = (0, 0, 2ab \sin\tfrac{\theta}{2} \cos\tfrac{\theta}{2}),$$
$$\mathbf{a}^{1/2} \cdot \mathbf{b}^{1/2} = a^{1/2} b^{1/2} \cos\tfrac{\theta}{2}, \text{ and}$$
$$2(\mathbf{a}^{1/2} \times \mathbf{b}^{1/2}) = (0, 0, 2a^{1/2} b^{1/2} \sin\tfrac{\theta}{2}),$$

so the second scalar times the third vector equals $\mathbf{a} \times \mathbf{b}$.

(c) Since $|\mathbf{a} - \mathbf{b}|^2 = a^2 + b^2 - 2ab\cos\theta \geq 2ab(1 - \cos\theta) = 4ab\sin^2(\frac{\theta}{2}) = 4|\mathbf{a}^{1/2} \times \mathbf{b}^{1/2}|^2$, it follows that $|\mathbf{a} - \mathbf{b}| \geq 2a^{1/2}b^{1/2}|\sin(\frac{\theta}{2})|$. The inequality arises from $a^2 + b^2 \geq 2ab$, and this becomes an equality if and only if $a = b$. We thus have (c). ■

Lemma 2.2. *Let $\mathbf{a} \neq \mathbf{b}$ be nonzero vectors, let O be the origin, and $\triangle OPQ$ a triangle such that $\mathbf{a} = \overrightarrow{OP}$ and $\mathbf{b} = \overrightarrow{OQ}$. The distance h of O to the line PQ satisfies the following:*

(d) $h = \dfrac{|\mathbf{a} \times \mathbf{b}|}{|\mathbf{a} - \mathbf{b}|} \leq |\mathbf{a}^{1/2} \cdot \mathbf{b}^{1/2}|$, with equality holding if and only if $|\mathbf{a}| = |\mathbf{b}|$.

We remark that one is tempted to just call h "the distance from O to $\mathbf{a} - \mathbf{b}$."

Proof. The area of the parallelogram spanned by $\mathbf{a}$ and $\mathbf{b}$ is $|\mathbf{a} \times \mathbf{b}|$, which equals $|\mathbf{a} - \mathbf{b}| \cdot h$. The equality is therefore clear. Next, by (b) and (c) in Lemma 2.1, we have

$$h = 2|\mathbf{a}^{1/2} \cdot \mathbf{b}^{1/2}||\mathbf{a}^{1/2} \times \mathbf{b}^{1/2}|/|\mathbf{a} - \mathbf{b}| \leq |\mathbf{a}^{1/2} \cdot \mathbf{b}^{1/2}|.$$

Equality holds if and only if (c) is an equality, which happens if and only if $|\mathbf{a}| = |\mathbf{b}|$. ■

3. PROOF OF ERDŐS' INEQUALITY. Using the previously defined notations, we further denote that

$$\overrightarrow{OA} = \mathbf{p} = (p\cos\theta_p, p\sin\theta_p, 0),$$

$$\overrightarrow{OB} = \mathbf{q} = (q\cos\theta_q, q\sin\theta_q, 0),$$

and

$$\overrightarrow{OC} = \mathbf{r} = (r\cos\theta_r, r\sin\theta_r, 0).$$

Also, without loss of generality, we may and shall assume in the rest of the paper that $\mathbf{O} = (0, 0, 0)$, $\mathbf{A} = (p, 0, 0)$, $\mathbf{B} = (b_1, b_2, 0)$ with $b_2 > 0$, and $\mathbf{C} = (c_1, c_2, 0)$ with $c_2 < 0$. It follows that

$$\theta_p = \angle AOA = 0, \; 0 < \theta_q = \angle AOB < \pi,$$

and

$$\pi < \theta_r = \angle AOC < 2\pi,$$

so that

$$0 < \theta_q - \theta_p < \pi, \; 0 < \theta_r - \theta_q < \pi, \;\; \text{and}\, \pi < \theta_r - \theta_p < 2\pi. \tag{2}$$

We now turn to proving Erdős' inequality. From Lemma 2.1(a), Lemma 2.2(d) and (2) above, we have

$$\begin{aligned} p + q + r - 2(u + v + w) & \\ \geq \mathbf{p}^{1/2} \cdot \mathbf{p}^{1/2} + \mathbf{q}^{1/2} \cdot \mathbf{q}^{1/2} + \mathbf{r}^{1/2} \cdot \mathbf{r}^{1/2} & \\ -2\mathbf{q}^{1/2} \cdot \mathbf{r}^{1/2} + 2\mathbf{p}^{1/2} \cdot \mathbf{r}^{1/2} - 2\mathbf{p}^{1/2} \cdot \mathbf{q}^{1/2} & \\ = |\mathbf{p}^{1/2} - \mathbf{q}^{1/2} + \mathbf{r}^{1/2}|^2 \geq 0. & \end{aligned} \tag{3}$$

It follows that

$$p + q + r \geq 2(u + v + w), \tag{4}$$

which is Erdős' inequality, as desired. ■

Next, we establish the condition for the equality to hold in (4) (i.e., for the two inequalities in (3) to both be equalities). The first of the two inequalities in (3) is due to the three inequalities $u \leq |\mathbf{q}^{1/2} \cdot \mathbf{r}^{1/2}|$, $v \leq |\mathbf{p}^{1/2} \cdot \mathbf{r}^{1/2}|$ and $w \leq |\mathbf{p}^{1/2} \cdot \mathbf{q}^{1/2}|$. These three all depend upon Lemma 2.2(d), and the equality condition there implies $p = q = r$. We therefore conclude that the first inequality of (3) is an equality if and only if vertices A, B, C lie on a circle centered at O. Let us assume that this condition holds for the remainder of this note.

Now consider the second inequality of (3). Clearly, this becomes an equality if and only if $\mathbf{q}^{1/2} = \mathbf{p}^{1/2} + \mathbf{r}^{1/2}$, where $|\mathbf{p}^{1/2}| = |\mathbf{q}^{1/2}| = |\mathbf{r}^{1/2}|$ and $\arg(\mathbf{p}^{1/2}) = 0$. Hence, one sees at once that $\mathbf{p}^{1/2}$ and $\mathbf{r}^{1/2}$ generate a parallelogram with all four sides equal, where $\mathbf{q}^{1/2}$ is its diagonal, also of equal size. We conclude that the condition for the equality in (4) is that tips A, B, and C of the vectors $\mathbf{p}$, $\mathbf{q}$, $\mathbf{r}$ rooted at O form an equilateral triangle with its center at O.

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REFERENCES

1. C. Alsina, R. B. Nelson, A Visual Proof of the Erdős-Mordell Inequality, *Forum Geom.* **7** (2007) 99–107.
2. A. Avez, A short proof of a theorem of Erdős and Mordell, *Amer. Math. Monthly* **100** (1993) 60–62; available at http://dx.doi.org/10.2307/2324817.
3. D. K. Kazarinoff, A simple proof of the Erdős-Mordell Inequallity for triangles, *Michigan Math. J.* **4** (1957) 97–98; available at http://dx.doi.org/10.1307/mmj/1028988998.
4. Kusco, A proof of the Erdős inequality, *Sugaku Seminar*, March issue (2008) 51–52 (IN JAPANESE).
5. A. Sakurai, A proof by using vector analysis to the Erdős Inequality on a triangle, *Transactions of Math. Education for Colleges & Univ.*, **17** (2010) 11–14 (IN JAPANESE).

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