An Elementary Counterexample in the Compact-Open Topology
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which again may alternatively be proved by using quadratic reciprocity and the known value of \( \left( \frac{2}{p} \right) \).

To conclude, we show that our main theorem also holds for negative values of \( a \) and \( b \). First, if \( a, b \) are any integers then

\[
\left\lceil \frac{a}{2} \right\rceil \left\lfloor \frac{b}{2} \right\rfloor - \left\lceil \frac{-a}{2} \right\rceil \left\lfloor \frac{-b}{2} \right\rfloor \tag{2}
\]

is even if \( ab \) is a multiple of 4, odd if \( ab \) is a multiple of 2 but not of 4. To see this write

\[ a = 2c + x, \quad b = 2d + y. \]

where \( c, d \) are integers and \( x, y \in \{0, 1\} \). Then the expression (2) is

\[ (c + x)d - (-c)(-d - y) = dx - cy. \]

If \( ab \) is a multiple of 4 then either \( x = y = 0 \); or \( x = 0, c \) is even; or \( y = 0, d \) is even. In each case (2) is even. If \( ab \) is a multiple of 2 but not of 4, then either \( x = 0, c \) is odd, \( y = 1 \); or \( x = 1, d \) is odd, \( y = 0 \). In each case (2) is odd, and our first claim is proved. Consequently, if \( a, b \) are negative and \( ab = p - 1 \), then

\[
\left\lceil \frac{a}{2} \right\rceil \left\lfloor \frac{b}{2} \right\rfloor - \left\lceil \frac{-a}{2} \right\rceil \left\lfloor \frac{-b}{2} \right\rfloor \quad \text{and} \quad \frac{p - 1}{2}
\]

have the same parity and so

\[
\left( \frac{b}{p} \right) = \left( \frac{-1}{p} \right) \left( \frac{-b}{p} \right) = (-1)^{p-1/2}(-1)^{\left\lceil-a/2\right\rceil \left\lceil-b/2\right\rceil} = (-1)^{\left\lceil a/2 \right\rceil \left\lceil b/2 \right\rceil},
\]

as we have already shown for positive \( a \) and \( b \).

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Jonathan Groves

**Abstract.** We give a short proof that the space of continuous functions from \([0, 1]\) to \([0, 1]\) is not compact in the compact-open topology.

Suppose \( X \) and \( Y \) are compact topological spaces. Let \( C(X, Y) \) be the space of continuous functions from \( X \) to \( Y \), and give this space the compact-open topology. An interesting problem from topology is to prove or disprove that \( C(X, Y) \) is compact.
What is the compact-open topology on \( C(X, Y) \)? Let \( C \) be a compact subset of \( X \) and \( U \) an open subset of \( Y \). Let \( S(C, U) \) be the set of all functions \( f \in C(X, Y) \) such that \( f(C) \subseteq U \). Then the sets \( S(C, U) \) form a subbasis of the compact-open topology on \( C(X, Y) \).

It turns out that \( C(X, Y) \) need not be compact even if \( X \) and \( Y \) are. This is known to experts, but not found in elementary texts such as [1], [2], and [3]. The purpose of this note is to provide an elementary counterexample; all we need is the intermediate value theorem. In our counterexample, we let \( X = Y = I \), the closed unit interval \([0, 1]\) with the usual subspace topology inherited from \( \mathbb{R} \). A common proof that \( C(I, I) \) is not compact notes that the compact-open topology agrees with the uniform topology on \( C(I, I) \) and that the sequence \((f_n)\) defined by \( f_n(x) = x^n \) has no uniformly convergent subsequence since the limiting function is not continuous.

For our proof, pick \( \epsilon < 1/2 \). For \( x \in I \), let \( U_x = S(\{x\}, (x - \epsilon, x + \epsilon) \cap I) \). These sets form an open cover of \( C(I, I) \) because, by the intermediate value theorem, every continuous function from \( I \) to \( I \) has a fixed point. We now prove that this open cover has no finite subcover. Let \( U_{x_1}, U_{x_2}, \ldots, U_{x_n} \) be a finite subcollection of this open cover and, without loss of generality, assume \( x_1 < x_2 < \cdots < x_n \). Since \( \epsilon < 1/2 \), no set \( U_{x_i} \) covers \( C(I, I) \). Choose \( y_i \in I \setminus (x_i - \epsilon, x_i + \epsilon) \) for all \( i = 1, 2, \ldots, n \). Let \( f \) be the piecewise linear function connecting \((0, f(0)), (x_1, y_1), (x_2, y_2), \ldots, (x_n, y_n)\), and \((1, f(1))\), where \( f(0) \) is taken to be 0 if \( x_1 \neq 0 \) and \( f(1) \) is taken to be 1 if \( x_n \neq 1 \). Then it is clear that \( f \notin U_{x_i} \) for all \( i \), but \( f \in C(I, I) \), which proves that this finite subcollection does not cover \( C(I, I) \). Thus, \( C(I, I) \) is not compact in the compact-open topology.

I like this proof because it is a good illustration of the definitions of compactness and the compact open topology, and is a good application of the intermediate value theorem. A comparison of both this proof and the more common proof should be valuable to students.

**REFERENCES**


**JONATHAN GROVES** held a bachelors degree from Austin Peay State University and masters degrees from Western Kentucky University and the University of Kentucky, and was employed by Kaplan University. 
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**Editor’s Note:** Jonathan Groves passed away on June 4, 2011 at the age of 29 before this note was accepted. The MONTHLY thanks a colleague of Jonathan’s, who wishes to remain anonymous, who saw the note through revisions and proofs. We extend our deepest condolences to Jonathan’s family.