which again may alternatively be proved by using quadratic reciprocity and the known value of $\left(\frac{2}{p}\right)$.

To conclude, we show that our main theorem also holds for negative values of a and b. First, if a, b are any integers then

$$\left\lceil\frac{a}{2}\right\rceil\left\lfloor\frac{b}{2}\right\rfloor-\left\lceil\frac{-a}{2}\right\rceil\left\lfloor\frac{-b}{2}\right\rfloor \tag{2}$$

is even if ab is a multiple of 4, odd if ab is a multiple of 2 but not of 4. To see this write

$$a=2c+x, \quad b=2d+y,$$

where c, d are integers and $x, y \in \{0, 1\}$. Then the expression (2) is

$$(c+x)d-(-c)(-d-y)=dx-cy.$$

If ab is a multiple of 4 then either $x=y=0$; or $x=0$, c is even; or $y=0$, d is even. In each case (2) is even. If ab is a multiple of 2 but not of 4, then either $x=0$, c is odd, $y=1$; or $x=1$, d is odd, $y=0$. In each case (2) is odd, and our first claim is proved. Consequently, if a, b are negative and $ab=p-1$, then

$$\left\lceil\frac{a}{2}\right\rceil\left\lfloor\frac{b}{2}\right\rfloor-\left\lceil\frac{-a}{2}\right\rceil\left\lfloor\frac{-b}{2}\right\rfloor \quad \text{and} \quad \frac{p-1}{2}$$

have the same parity and so

$$\left(\frac{b}{p}\right)=\left(\frac{-1}{p}\right)\left(\frac{-b}{p}\right)=(-1)^{(p-1)/2}(-1)^{\lceil -a/2\rceil\lfloor -b/2\rfloor}=(-1)^{\lceil a/2\rceil\lfloor b/2\rfloor},$$

as we have already shown for positive a and b.

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An Elementary Counterexample in the Compact-Open Topology

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Abstract. We give a short proof that the space of continuous functions from [0, 1] to [0, 1] is not compact in the compact-open topology.

Suppose X and Y are compact topological spaces. Let $\mathcal{C}(X, Y)$ be the space of continuous functions from X to Y, and give this space the compact-open topology. An interesting problem from topology is to prove or disprove that $\mathcal{C}(X, Y)$ is compact.

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What is the compact-open topology on $\mathcal{C}(X, Y)$? Let C be a compact subset of X and U an open subset of Y. Let $S(C, U)$ be the set of all functions $f \in \mathcal{C}(X, Y)$ such that $f(C) \subset U$. Then the sets $S(C, U)$ form a subbasis of the compact-open topology on $\mathcal{C}(X, Y)$.

It turns out that $\mathcal{C}(X, Y)$ need not be compact even if X and Y are. This is known to experts, but not found in elementary texts such as [**1**], [**2**], and [**3**]. The purpose of this note is to provide an elementary counterexample; all we need is the intermediate value theorem. In our counterexample, we let $X = Y = I$, the closed unit interval [0, 1] with the usual subspace topology inherited from $\mathbb{R}$. A common proof that $\mathcal{C}(I, I)$ is not compact notes that the compact-open topology agrees with the uniform topology on $\mathcal{C}(I, I)$ and that the sequence (f_n) defined by $f_n(x) = x^n$ has no uniformly convergent subsequence since the limiting function is not continuous.

For our proof, pick $\epsilon < 1/2$. For $x \in I$, let $U_x = S(\{x\}, (x - \epsilon, x + \epsilon) \cap I)$. These sets form an open cover of $\mathcal{C}(I, I)$ because, by the intermediate value theorem, every continuous function from I to I has a fixed point. We now prove that this open cover has no finite subcover. Let $U_{x_1}, U_{x_2}, \ldots, U_{x_n}$ be a finite subcollection of this open cover and, without loss of generality, assume $x_1 < x_2 < \cdots < x_n$. Since $\epsilon < 1/2$, no set U_{x_i} covers $\mathcal{C}(I, I)$. Choose $y_i \in I \setminus (x_i - \epsilon, x_i + \epsilon)$ for all $i = 1, 2, \ldots, n$. Let f be the piecewise linear function connecting $(0, f(0))$, (x_1, y_1), (x_2, y_2), $\ldots$, (x_n, y_n), and $(1, f(1))$, where $f(0)$ is taken to be 0 if $x_1 \neq 0$ and $f(1)$ is taken to be 1 if $x_n \neq 1$. Then it is clear that $f \notin U_{x_i}$ for all i, but $f \in \mathcal{C}(I, I)$, which proves that this finite subcollection does not cover $\mathcal{C}(I, I)$. Thus, $\mathcal{C}(I, I)$ is not compact in the compact-open topology.

I like this proof because it is a good illustration of the definitions of compactness and the compact open topology, and is a good application of the intermediate value theorem. A comparison of both this proof and the more common proof should be valuable to students.

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Editor's Note: Jonathan Groves passed away on June 4, 2011 at the age of 29 before this note was accepted. The MONTHLY thanks a colleague of Jonathan's, who wishes to remain anonymous, who saw the note through revisions and proofs. We extend our deepest condolences to Jonathan's family.