# Arithmetic Polygons

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**Abstract.** We consider the question of the existence of equiangular polygons with edge lengths in arithmetic progression, and show that they do not exist when the number of sides is a power of two and do exist if it is any other even number. A few results for small odd numbers are given.

An Olympiad problem [**1**] asked the following.

> Prove that there exists a convex 1990-gon with the following two properties:
>
> (a) all angles are equal.
>
> (b) the lengths of the 1990 sides are the numbers $1^2, 2^2, 3^2, \dots, 1990^2$ in some order.

Recently, remembering the question imperfectly, I constructed a convex 1990-gon with side lengths $1, 2, 3, \dots, 1990$, and wondered for which other $N$ such an $N$-gon could be constructed. This note gives a partial answer. Define an *arithmetic polygon* to be an equiangular polygon with edge lengths forming (upon suitable rearrangement) a nondegenerate arithmetic sequence.

**Lemma 1.** *For any $N$, if there exists an arithmetic $N$-gon, there exists one such with edge lengths* $1, 2, \dots, N$.

*Proof.* We work in the complex plane, so that an edge of length $L$ oriented at an angle $\theta$ to the positive real axis is represented by the complex number $Le^{i\theta}$. An equiangular $N$-gon has edge orientations (in cyclic order) $e^{2(j/N)\pi i}$, and edge lengths $a + p(j)b$ for some permutation $p : (0, 1, \dots, n-1) \to (0, 1, \dots, n-1)$. This polygonal path closes if and only if

$$\sum_{j=0}^{N-1}(a + p(j)b)e^{2(j/N)\pi i} = 0.$$

But we know (this is equivalent to the existence of *regular* $N$-gons) that

$$\sum_{j=0}^{N-1} e^{2(j/N)\pi i} = 0,$$

so

$$\sum_{j=0}^{N-1}(p(j) + 1)e^{2(j/N)\pi i} = 0. \qquad \blacksquare$$

http://dx.doi.org/10.4169/amer.math.monthly.119.08.695
MSC: Primary 52B12

**Theorem 2.** *For any even $N$ not a power of 2, and any positive real $a, b$, there exists a convex arithmetic $N$-gon with sides (in some order) $\{a, a+b, a+2b, \ldots, a+(N-1)b\}$.*

*Proof.* We first consider the case in which $N = 4k+2$ for some $k$. There is clearly a nonconvex arithmetic $(4k+2)$-gon in the complex plane with edges, in order,

$$\big(-(a+(2k+1)b), a, -(a+(2k+2)b)e^{2\pi i/(2k+1)}, (a+b)e^{2\pi i/(2k+1)}, \ldots,$$
$$-(a+(2k+j+1)b)e^{2j\pi i/(2k+1)}, (a+jb)e^{2j\pi i/(2k+1)}, \ldots,$$
$$-(a+(4k+1)b)e^{4k\pi i/(2k+1)}, (a+2kb)e^{4k\pi i/(2k+1)}\big)$$

(see Figure 1a).

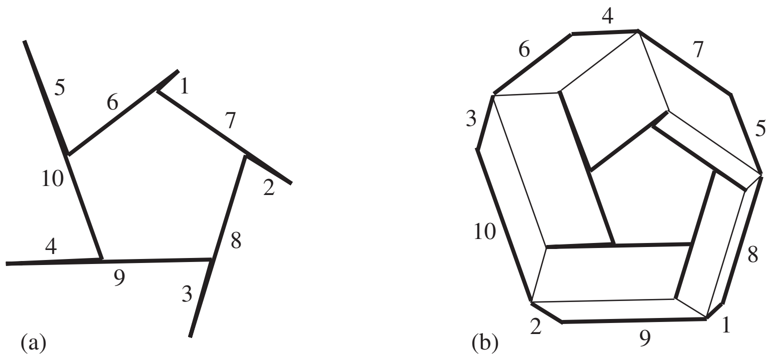


**Figure 1.** Construction of an arithmetic polygon with $4k+2$ sides.

To see that this closes, we group the edges in consecutive pairs, and note that

$$(a+(2k+j+1)b)e^{2j\pi i/(2k+1)} - (a+jb)e^{2j\pi i/(2k+1)} = (2k+1)be^{2j\pi i/(2k+1)}$$

and the values taken by the right-hand side are the sides of a regular $2k+1$-gon. As shown in Figure 1b, we can rearrange the edges to obtain a convex arithmetic $(4k+2)$-gon.

Any other even $N$ that is not a power of 2 has a factor of the form $4k+2$, $k > 0$. We break the arithmetic sequence $\{a, a+b, a+2b, \ldots, a+(m-1)b\}$ apart into $N/(4k+2)$ arithmetic sequences of length $4k+2$. For each of these, we construct a convex arithmetic polygon $P_j$, $j = 0, 1, \ldots, N/(4k+2)-1$. We rotate each $P_j$ by an angle of $2j\pi i/N$ to obtain $P'_j$, and construct a new polygon $\bigoplus_{j=0}^{N/(4k+2)-1} P'_j$ (the *Minkowski sum*, see for instance Moszynska [**2**, p. 66]) by interleaving the edges of the rotated polygons $P'_j$ in order of orientation (see Figure 2). ■

What about $N$-gons for which $N$ is either odd or a power of 2? By trying to sketch an arithmetic quadrilateral or octagon, the reader should be able to get most of the ideas motivating the following negative result.

**Theorem 3.** *There does not exist an equiangular $2^n$-gon with integer edge lengths, all distinct.*

*Proof.* We prove this by induction; clearly it is true for $n = 2$. Suppose that it is true for $n$, and let $P$ be a $2^{n+1}$-gon in the complex plane, with edges, in cyclic order, given

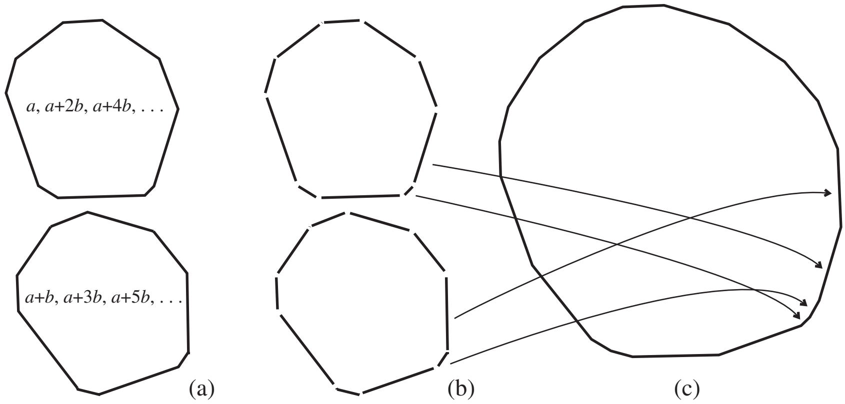


**Figure 2.** Interleaving edges of arithmetic $(4k+2)$-gons to obtain an arithmetic $((2k+1)2^n)$-gon.

by $\{P(k)e^{k\pi i/2^n} : 0 \le k < 2^n\}$, where $P(k) : \{1, \ldots, 2^n\} \to \mathbb{N}$ is one-to-one. Compute the sum of the even-numbered edges and the sum of the odd-numbered edges:

$$p_0 = \sum_{k=0}^{2^{n-1}} P(2k)e^{2k\pi i/2^n}, \quad p_1 = \sum_{k=0}^{2^{n-1}} P(2k+1)e^{(2k+1)\pi i/2^n}.$$

By hypothesis, neither $p_0$ nor $p_1$ can equal 0. Moreover, we have that

$$p_0 \in \mathbb{Q}[e^{\pi i/2^{n-1}}], \quad p_1 \in e^{\pi i/2^n}\mathbb{Q}[e^{\pi i/2^{n-1}}], \text{ and } \quad p_0 + p_1 = 0.$$

We would conclude that $e^{\pi i/2^n} \in \mathbb{Q}[e^{\pi i/2^{n-1}}]$, so that $\mathbb{Q}[e^{\pi i/2^n}] = \mathbb{Q}[e^{\pi i/2^{n-1}}]$. But this is well-known to be false (for instance, the dimension over $\mathbb{Q}$ of the first field is $2^{n-1}$ while that of the second is $2^{n-2}$.) Thus the theorem holds for $n+1$, and by induction for all $n \in \mathbb{N}$. ■

Lemma 1 gives us an immediate corollary.

**Corollary 3.1.** *There are no arithmetic* $2^n$*-gons.*

We are left with the case in which the number of sides is odd. It has been known since antiquity that every equiangular triangle is equilateral (this follows immediately from Euclid I.6); but this is easily seen not to be true for other equiangular $2k+1$-gons. For larger $k$, let consecutive edge lengths be $(a_{-k}, \ldots, a_{-1}, a_0, a_1, \ldots, a_k)$. Projecting onto a line perpendicular to edge 0 gives

$$\sum_{j=1}^{k} a_j \sin(2\pi j/(2k+1)) = \sum_{j=1}^{k} a_{-j} \sin(2\pi j/(2k+1)),$$

or

$$\sum_{j=1}^{k} (a_j - a_{-j}) \sin(2\pi j/(2k+1)) = 0.$$

**Proposition 4.** *Every equiangular pentagon with rational edges is regular.*

*Proof.* Let $d_i := a_i - a_{-i}$. For any choice of "edge 0" we have

$$d_2 \sin(4\pi/5) = -d_1 \sin(2\pi/5).$$

As $\sin(2\pi/5)/\sin(4\pi/5) = \tau$ is irrational, this can only be solved over the integers if $a_1 = a_{-1}$ and $a_2 = a_{-2}$. Letting each edge in turn be "edge 0," the conclusion follows. ■

By Lemma 1 we obtain the following.

**Corollary 4.1.** *There are no arithmetic pentagons.*

Extending the same idea yields the following.

**Proposition 5.** *Every equiangular heptagon with constructible edges is regular.*

*Proof.* For any choice of "edge 0" we have

$$d_1 \sin(2\pi/7) + d_2 \sin(4\pi/7) + d_3 \sin(6\pi/7) = 0,$$

which by the double- and triple-angle formulae reduces to

$$\sin(2\pi/7)[d_1 + 2d_2 \cos(2\pi/7) + d_3(4\cos^2(2\pi/7) - 1)] = 0,$$

so that

$$4d_3[\cos(2\pi/7)]^2 + 2d_2[\cos(2\pi/7)] + (d_1 - d_3) = 0.$$

But $\cos(2\pi/7)$ is well-known not to be constructible (see almost any senior undergraduate textbook on geometry or abstract algebra), and the set of constructible numbers is (essentially by definition) closed under degree 2 extensions; so we must have $d_1 = d_2 = d_3 = 0$. ■

**Corollary 5.1.** *There are no arithmetic heptagons.*

It is tempting to conjecture that the same results hold for any equiangular polygon with an odd number of sides. However, interleaving (for instance) the edges of three equilateral triangles of different edge lengths, two of them rotated by $\pm 20°$ with respect to the third, gives an equiangular but non-regular enneagon. We close with the following two conjectures.

**Conjecture 6.** *Every equiangular polygon with a prime number of edges, all rational, is regular.*

**Conjecture 7.** *No arithmetic $N$-gon exists for any odd $N$.*

**ACKNOWLEDGMENTS.** I would like to acknowledge financial support from an NSERC Discovery Grant, and the helpful suggestions of the anonymous referees.

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