Chapter 7

Use the following to answer questions 1-16:

  In the questions below, describe each sequence recursively. Include initial conditions and assume that the sequences begin with a_1.

1. $a_n = 5^n$.
   Ans: $a_n = 5a_{n-1}, a_1 = 5$.

2. The Fibonacci numbers.
   Ans: $a_n = a_{n-1} + a_{n-2}, a_1 = a_2 = 1$.

3. 0,1,0,1,0,1,....
   Ans: $a_n = a_{n-2}, a_1 = 0, a_2 = 1$.

4. $a_n = 1 + 2 + 3 + ... + n$.
   Ans: $a_n = a_{n-1} + n, a_1 = 1$.

5. 3,2,1,0,–1,–2,....
   Ans: $a_n = a_{n-1} - 1, a_1 = 3$.

6. $a_n = n!$.
   Ans: $a_n = na_{n-1}, a_1 = 1$.

7. 1/2,1/3,1/4,1/5,....
   Ans: $a_n = \dfrac{a_{n-1}}{1 + a_{n-1}}$, $a_1 = 1/2$.

8. 0.1, 0.11, 0.111, 0.1111,....
   Ans: $a_n = a_{n-1} + 1/10^n, a_1 = 0.1$.

9. $1^2, 2^2, 3^3, 4^2,....$
   Ans: $a_n = a_{n-1} + 2n - 1, a_1 = 1$.

10. 1,111,11111,1111111,....
    Ans: $a_n = 100a_{n-1} + 11$.

11. $a_n =$ the number of subsets of a set of size n.
    Ans: $a_n = 2 \cdot a_{n-1}, a_1 = 2$.

12. 1,101,10101,1010101,....
    Ans: $a_n = 100a_{n-1} + 1, a_1 = 1$.

13. a_n = the number of bit strings of length n with an even number of 0s.
   Ans: $a_n = a_{n-1} + 2^{n-2}$, $a_1 = 1$.

14. a_n = the number of bit strings of length n that begin with 1.
   Ans: $a_n = 2a_{n-1}$, $a_1 = 1$.

15. a_n = the number of bit strings of length n that contain a pair of consecutive 0s.
   Ans: $a_n = a_{n-1} + a_{n-2} + 2^{n-2}$, $a_1 = 0$, $a_2 = 1$.

16. a_n = the number of ways to go down an n-step staircase if you go down 1, 2, or 3 steps at a time.
   Ans: $a_n = a_{n-1} + a_{n-2} + a_{n-3}$, $a_1 = 0$, $a_2 = 1$, $a_3 = 1$.

17. Verify that $a_n = 6$ is a solution to the recurrence relation $a_n = 4a_{n-1} - 3a_{n-2}$.
   Ans: $4 \cdot 6 - 3 \cdot 6 = 1 \cdot 6 = 6$.

18. Verify that $a_n = 3^n$ is a solution to the recurrence relation $a_n = 4a_{n-1} - 3a_{n-2}$.
   Ans: $4 \cdot 3^{n-1} - 3 \cdot 3^{n-2} = 4 \cdot 3^{n-1} - 3^{n-1} = 3 \cdot 3^{n-1} = 3^n$.

19. Verify that $a_n = 3^{n+4}$ is a solution to the recurrence relation $a_n = 4a_{n-1} - 3a_{n-2}$.
   Ans: $4 \cdot 3^{n+3} - 3 \cdot 3^{n+2} = 4 \cdot 3^{n+3} - 3^{n+3} = 3 \cdot 3^{n+3} = 3^{n+4}$.

20. Verify that $a_n = 3^n + 1$ is a solution to the recurrence relation $a_n = 4a_{n-1} - 3a_{n-2}$.
   Ans: $4(3^{n-1} + 1) - 3(3^{n-2} + 1) = 4 \cdot 3^{n-1} - 3^{n-1} + 4 - 3 = 3^{n-1}(4 - 1) + 1 = 3^n + 1$.

21. Verify that $a_n = 7 \cdot 3^n - \pi$ is a solution to the recurrence relation $a_n = 4a_{n-1} - 3a_{n-2}$.
   Ans: $4(7 \cdot 3^{n-1} - \pi) - 3(7 \cdot 3^{n-2} - \pi) = 28 \cdot 3^{n-1} - 7 \cdot 3^{n-1} - 4\pi + 3\pi = 7 \cdot 3^n - \pi$.

Use the following to answer questions 22-26:

 In the questions below find a recurrence relation with initial condition(s) satisfied by the sequence. Assume a_0 is the first term of the sequence.

22. $a_n = 2^n$.
   Ans: $a_n = 2a_{n-1}$, $a_0 = 1$.

23. $a_n = 2^n + 1$.
   Ans: $a_n = 2a_{n-1} - 1$, $a_0 = 2$.

24. $a_n = (-1)^n$.
   Ans: $a_n = -a_{n-1}$, $a_0 = 1$.

25. $a_n = 3n - 1$.
   Ans: $a_n = a_{n-1} + 3$, $a_0 = -1$.

26. $a_n = \sqrt{2}$.

Ans: $a_n = a_{n-1}$, $a_0 = \sqrt{2}$.

27. You take a job that pays $25,000 annually.
   (a) How much do you earn n years from now if you receive a three percent raise each year?
   (b) How much do you earn n years from now if you receive a five percent raise each year?
   (c) How much do you earn n years from now if each year you receive a raise of $1000 plus two percent of your previous year's salary.
   Ans: (a) $25,000 \cdot 1.03^n$. (b) $25,000 \cdot 1.05^n$. (c) $25,000 \cdot 1.02^n + 1,000(\frac{1.02^n - 1}{0.02})$.

28. Suppose inflation continues at three percent annually. (That is, an item that costs $1.00 now will cost $1.03 next year.) Let a_n = the value (that is, the purchasing power) of one dollar after n years.
   (a) Find a recurrence relation for a_n.
   (b) What is the value of $1.00 after 20 years?
   (c) What is the value of $1.00 after 80 years?
   (d) If inflation were to continue at ten percent annually, find the value of $1.00 after 20 years.
   (e) If inflation were to continue at ten percent annually, find the value of $1.00 after 80 years.
   Ans: (a) $a_n = a_{n-1}/1.03$. (b) $a_{20} = 1/1.03^{20} \approx 0.55$. (c) $a_{80} = 1/1.03^{80} \approx 0.09$. (d) $1/1.1^{20} \approx 0.15$. (e) $1/1.1^{80} \approx 0.00$.

Use the following to answer questions 29-34:

  In the questions below determine whether the recurrence relation is a linear homogeneous recurrence relation with constant coefficients.

29. $a_n = 0.7a_{n-1} - 0.3a_{n-2}$.
   Ans: Yes.

30. $a_n = na_{n-1}$.
   Ans: No.

31. $a_n = 5a_{n-1}^2 - 3a_{n-2}^2$.
   Ans: No.

32. $a_n = a_{n-3}$.
   Ans: Yes.

33. $a_n - 7a_{n-2} + a_{n-5} = 0$.
    Ans: Yes.

34. $a_n + a_{n-1} = 1$.
    Ans: No.

35. A vending machine dispensing books of stamps accepts only $1 coins, $1 bills, and $2 bills. Let a_n denote the number of ways of depositing n dollars in the vending machine, where the order in which the coins and bills are deposited matters.
    (a) Find a recurrence relation for a_n and give the necessary initial condition(s).
    (b) Find an explicit formula for a_n by solving the recurrence relation in part (a).
    Ans: (a) $a_n = 2a_{n-1} + a_{n-2}$, $a_0 = 1$, $a_1 = 2$. (b) $a_n = \alpha\left(1+\sqrt{2}\right)^n + \beta\left(1-\sqrt{2}\right)^n$ where
    $\alpha = \left(1 + \frac{1}{\sqrt{2}}\right)\big/2$ and $\beta = (1 - \frac{1}{\sqrt{2}})\big/2$.

36. Find the solution of the recurrence relation $a_n = 3a_{n-1}$ with $a_0 = 2$.
    Ans: $a_n = 2 \cdot 3^n$

Use the following to answer questions 37-45:

  In the questions below solve the recurrence relation either by using the characteristic equation or by discovering a pattern formed by the terms.

37. $a_n = 5a_{n-1} - 4a_{n-2}$, $a_0 = 1$, $a_1 = 0$.
    Ans: $a_n = (-1/3) \cdot 4^n + (4/3) \cdot 1^n$.

38. $a_n = 5a_{n-1} - 4a_{n-2}$, $a_0 = 0$, $a_1 = 1$.
    Ans: $a_n = (1/3) \cdot 4^n - (1/3) \cdot 1^n$.

39. $a_n = -10a_{n-1} - 21a_{n-2}$, $a_0 = 2$, $a_1 = 1$.
    Ans: $a_n = (-7/4)(-7)^n + (15/4) \cdot (-3)^n$.

40. $a_n = a_{n-2}$, $a_0 = 2$, $a_1 = -1$.
    Ans: $a_n = (1/2) \cdot 1^n + (3/2) \cdot (-1)^n$.

41. $a_n = 2a_{n-1} + 2a_{n-2}$, $a_0 = 0$, $a_1 = 1$.
    Ans: $a_n = \left(\sqrt{3}/6\right)\left(1+\sqrt{3}\right)^n - \left(\sqrt{3}/6\right)\left(1-\sqrt{3}\right)^n$.

42. $a_n = 3na_{n-1}$, $a_0 = 2$.
    Ans: $a_n = 2 \cdot 3^n \cdot n!$.

43. $a_n = a_{n-1} + 3n$, $a_0 = 5$.
   Ans: $a_n = 5 + 3\frac{n(n+1)}{2}$.

44. $a_n = 2a_{n-1} + 5$, $a_0 = 3$.
   Ans: $a_n = 3 \cdot 2^n + 5(2^n - 1) = 2^{n+3} - 5$.

45. $a_n = a_{n-1} + 2n + 1$, $a_0 = 5$.
   Ans: $a_n = 5 + n(n + 1) + n = n^2 + 2n + 5$.

46. The solutions to $a_n = -3a_{n-1} + 18a_{n-2}$ have the form $a_n = c \cdot 3^n + d \cdot (-6)^n$. Which of the following are solutions to the given recurrence relation?
   (a) $a_n = 3^{n+1} + (-6)^n$.
   (b) $a_n = 5(-6)^n$.
   (c) $a_n = 3c - 6d$.
   (d) $a_n = 3^{n-2}$.
   (e) $a_n = \pi(3^n + (-6)^n)$.
   (f) $a_n = -3^n$.
   (g) $a_n = 3^n(1 + (-2)^n)$.
   (h) $a_n = 3^n + 6^n$.
   Ans: (a) Yes. (b) Yes. (c) No. (d) Yes. (e) Yes. (f) Yes. (g) Yes. (h) No.

47. Assume that the characteristic equation for a homogeneous linear recurrence relation with constant coefficients is $(r - 5)^3 = 0$. Describe the form for the general solution to the recurrence relation.
   Ans: $a_n = c5^n + dn5^n + en^2 5^n$.

48. Assume that the characteristic equation for a homogeneous linear recurrence relation with constant coefficients is $(r + 2)(r + 4)^2 = 0$. Describe the form for the general solution to the recurrence relation.
   Ans: $a_n = c(-2)^n + d(-4)^n + en(-4)^n$.

49. Assume that the characteristic equation for a homogeneous linear recurrence relation with constant coefficients is $(r + 1)^4(r - 1)^4 = 0$. Describe the form for the general solution to the recurrence relation.
   Ans: $a_n = c(-1)^n + dn(-1)^n + en^2(-1)^n + fn^3(-1)^n + g + hn + in^2 + jn^3$.

50. Assume that the characteristic equation for a homogeneous linear recurrence relation with constant coefficients is $(r - 3)^2(r - 4)^3(r + 7)^2 = 0$. Describe the form for the general solution to the recurrence relation.
   Ans: $a_n = c3^n + dn3^n + en^2 3^n + f4^n + gn4^n + hn^2 4^n + i(-7)^n + jn(-7)^n$.

51. The Catalan numbers C_n also count the number of strings of n +'s and n −'s with the following property: as each string is read from left to right, the number of +'s encountered is always at least as large as the number of −'s.
(a) Verify this by listing these strings of lengths 2, 4, and 6 and showing that there are C_1, C_2, and C_3 of these, respectively.
(b) Explain how counting these strings is the same as counting the number of ways to correctly parenthesize strings of variables.
Ans: (a) C_1: + −, C_2: + − + −, + + − −, C_3: + − + − + −, + + + − − −, + + − + − −,
 + + − − + −, + − + + − −. (b) Treat each + as a left parenthesis and each − as a right parenthesis.

52. What form does a particular solution of the linear nonhomogeneous recurrence relation $a_n = 4a_{n-1} - 4a_{n-2} + F(n)$ have when $F(n) = 2^n$?
Ans: $p_0 n^2 2^n$.

53. What form does a particular solution of the linear nonhomogeneous recurrence relation $a_n = 4a_{n-1} - 4a_{n-2} + F(n)$ have when $F(n) = n2^n$?
Ans: $n^2(p_1 n + p_0)2^n$.

54. What form does a particular solution of the linear nonhomogeneous recurrence relation $a_n = 4a_{n-1} - 4a_{n-2} + F(n)$ have when $F(n) = n^2 \cdot 4^n$?
Ans: $(p_2 n^2 + p_1 n + p_0)4^n$.

55. What form does a particular solution of the linear nonhomogeneous recurrence relation $a_n = 4a_{n-1} - 4a_{n-2} + F(n)$ have when $F(n) = (n^2 + 1)2^n$?
Ans: $n^2\left(p_2 n^2 + p_1 n + p_0\right)2^n$.

56. Consider the recurrence relation $a_n = 2a_{n-1} + 3n$.
(a) Write the associated homogeneous recurrence relation.
(b) Find the general solution to the associated homogeneous recurrence relation.
(c) Find a particular solution to the given recurrence relation.
(d) Write the general solution to the given recurrence relation.
(e) Find the particular solution to the given recurrence relation when $a_0 = 1$.
Ans: (a) $a_n = 2a_{n-1}$. (b) $a_n = c2^n$. (c) $a_n = -3n - 6$. (d) $a_n = -3n - 6 + c2^n$. (e) $a_n = -3n - 6 + 7 \cdot 2^n$.

57. Consider the recurrence relation $a_n = -a_{n-1} + n$.
    (a) Write the associated homogeneous recurrence relation.
    (b) Find the general solution to the associated homogeneous recurrence relation.
    (c) Find a particular solution to the given recurrence relation.
    (d) Write the general solution to the given recurrence relation.
    (e) Find the particular solution to the given recurrence relation when $a_0 = 1$.
    Ans: (a) $a_n = -a_{n-1}$. (b) $a_n = c(-1)^n$. (c) $a_n = \frac{n}{2} + \frac{1}{4}$. (d) $a_n = \frac{n}{2} + \frac{1}{4} + c(-1)^n$. (e)
        $a_n = \frac{n}{2} + \frac{1}{4} + \frac{3}{4}(-1)^n$.

58. Consider the recurrence relation $a_n = 3a_{n-1} + 5^n$.
    (a) Write the associated homogeneous recurrence relation.
    (b) Find the general golution to the associated homogeneous recurrence relation.
    (c) Find a particular solution to the given recurrence relation.
    (d) Write the general solution to the given recurrence relation.
    (e) Find the particular solution to the given recurrence relation when $a_0 = 1$.
    Ans: (a) $a_n = 3a_{n-1}$ (b) $a_n = c3^n$ (c) $a_n = \dfrac{5^{n+1}}{2}$ (d) $a_n = \dfrac{5^{n+1}}{2} + c3^n$ (e)
        $a_n = \dfrac{5^{n+1}}{2} - \dfrac{3^{n+1}}{2}$

59. Consider the recurrence relation $a_n = 2a_{n-1} + 1$.
    (a) Write the associated homogeneous recurrence relation.
    (b) Find the general golution to the associated homogeneous recurrence relation.
    (c) Find a particular solution to the given recurrence relation.
    (d) Write the general solution to the given recurrence relation.
    (e) Find the particular solution to the given recurrence relation when $a_0 = 1$.
    Ans: (a) $a_n = 2a_{n-1}$. (b) $a_n = c2^n$. (c) $a_n = -1$. (d) $a_n = c2^n - 1$. (e) $a_n = 2^{n+1} - 1$.

60. Suppose $f(n) = 3f(n/2) + 1, f(1) = 1$. Find $f(8)$.
    Ans: 40.

61. Suppose $f(n) = f(n/3) + 2n, f(1) = 1$. Find $f(27)$.
    Ans: 79.

62. Suppose $f(n) = 2f(n/2)$, $f(8) = 2$. Find $f(1)$.
    Ans: 1/4.

63. Suppose $f(n) = 2f(n/2) + 3$, $f(16) = 51$. Find $f(2)$.
    Ans: 15/4.

64. Suppose $f(n) = 4f(n/2) + n + 2$, $f(1) = 2$. Find $f(8)$.
    Ans: 226.

65. Use generating functions to solve $a_n = 3a_{n-1} + 2^n$, $a_0 = 5$.
    Ans: $a_n = 7 \cdot 3^n - 2 \cdot 2^n$.

66. Use generating functions to solve $a_n = 5a_{n-1} + 1$, $a_0 = 1$.
    Ans: $a_n = \frac{5^{n+1}}{4} - \frac{1}{4}$.

Use the following to answer questions 67-76:

 In the questions below write the first seven terms of the sequence determined by the generating function.

67. $(x+3)^2$.
    Ans: (a) 9,6,1,0,0,0,0.

68. $(1+x)^5$.
    Ans: 1,5,10,10,5,1,0.

69. $(1+x)^9$.
    Ans: 1,9,36,84,126,126,84.

70. $\dfrac{1}{1-3x}$.
    Ans: 1,3,9,27,81,243,729.

71. $\dfrac{x^2}{1-x}$.
    Ans: 0,0,1,1,1,1,1.

72. $\dfrac{1+x}{1-x}$.
    Ans: 1,2,2,2,2,2,2.

73. 5.
    Ans: 5,0,0,0,0,0,0.

74. $e^x + e^{-x}$.
    Ans: $2,0,1,0,\frac{1}{12},0,\frac{1}{360}$.

75. $\cos x$.
    Ans: $1,0,\frac{-1}{2!},0,\frac{1}{4!},0,\frac{-1}{6!}$.

76. $\dfrac{1}{1-x} - x^2 - x^3$.

Ans: 1,1,0,0,1,1,1.

Use the following to answer questions 77-87:

In the questions below find the coefficient of x^8 in the power series of each of the function.

77. $\left(1 + x^2 + x^4\right)^3$.

Ans: 6.

78. $\left(1 + x^2 + x^4 + x^6\right)^3$.

Ans: 12.

79. $\left(1 + x^2 + x^4 + x^6 + x^8\right)^3$.

Ans: 15.

80. $\left(1 + x^2 + x^4 + x^6 + x^8 + x^{10}\right)^3$.

Ans: 15.

81. $(1 + x^3)^{12}$.

Ans: 0.

82. $\left(1 + x\right)\left(1 + x^2\right)\left(1 + x^3\right)\left(1 + x^4\right)\left(1 + x^5\right)$.

Ans: 3.

83. $\dfrac{1}{1-2x}$.

Ans: 2^8.

84. $\dfrac{x^3}{1-3x}$.

Ans: 3^5.

85. $\dfrac{1}{(1-x)^2}$.

Ans: 9.

86. $\dfrac{x^2}{(1+2x)^2}$.

Ans: $7 \cdot 2^6$.

87. $\frac{1}{1-3x^2}$.

  Ans: 3^4.

Use the following to answer questions 88-100:

In the questions below find a closed form for the generating function for the sequence.

88. 4,8,16,32,64,....

  Ans: $\dfrac{4}{1-2x}$.

89. 1,0,1,0,1,0,1,0,....

  Ans: $\dfrac{1}{1-x^2}$.

90. 2,0,0,2,0,0,2,0,0,2,....

  Ans: $\dfrac{2}{1-x^3}$.

91. 2,4,6,8,10,12,....

  Ans: $\dfrac{2}{(1-x)^2}$.

92. 0,0,0,1,1,1,1,0,0,0,0,0,0,0,0,0,....

  Ans: $x^3\left(1+x+x^2+x^3\right)$.

93. 2,3,4,5,6,7,....

  Ans: $\dfrac{1}{(1-x)^2}+\dfrac{1}{1-x}=\dfrac{2-x}{(1-x)^2}$.

94. 0,1,1,0,1,1,0,1,1,0,1,1,0,1....

  Ans: $\dfrac{1}{1-x}-\dfrac{1}{1-x^3}$.

95. $1,-1,\frac{1}{2!},-\frac{1}{3!},\frac{1}{4!},-\frac{1}{5!},\dots$.

  Ans: e^{-x}.

96. $1,\frac{1}{2!},\frac{1}{4!},\frac{1}{6!},\frac{1}{8!}\dots$.

  Ans: e^{n^2} .

97. $1,-1,1,-1,1,-1,1,-1,....$

Ans: $\dfrac{1}{1+x}$.

98. $1,0,-1,0,1,0,-1,0,1,0,-1,....$

Ans: $\dfrac{1}{1+x^2}$.

99. $\binom{50}{50},\binom{50}{49},\binom{50}{48},...,\binom{50}{1},\binom{50}{0},0,0,0,....$.

Ans: $(1+x)^{50}$.

100. $\binom{50}{1},2\binom{50}{2},3\binom{50}{3},...,50\binom{50}{50},0,0,0,....$.

Ans: $50(1+x)^{49}$.

101. Set up a generating function and use it to find the number of ways in which eleven identical coins can be put in three distinct envelopes if each envelope has at least two coins in it.

Ans: $(x^2 + x^3 + x^4 + ...)^3$, 21.

102. Set up a generating function and use it to find the number of ways in which eleven identical coins can be put in three distinct envelopes if each envelope has most six coins in it.

Ans: $(1 + x + x^2 + ... + x^6)^3$, 33.

103. Set up a generating function and use it to find the number of ways in which eleven identical coins can be put in three distinct envelopes if no envelope is empty.

Ans: $(x + x^2 + x^3 + ...)^3$, 45.

104. Set up a generating function and use it to find the number of ways in which eleven identical coins can be put in three distinct envelopes if each envelope has an even number of coins in it.

Ans: $(1 + x^2 + x^4 + x^6 + ...)^3$, 0.

105. Set up a generating function and use it to find the number of ways in which eleven identical coins can be put in three distinct envelopes if each envelope has at least two but no more than five coins in it.

Ans: $(x^2 + x^3 + x^4 + x^5)^3$, 12.

106. Set up a generating function and use it to find the number of ways in which eleven identical coins can be put in three distinct envelopes (labeled A, B, C) if envelope A has at least three coins in it.
Ans: $(x^3 + x^4 + x^5 + x^6 + ...)(1 + x + x^2 + x^3 + ...)^2$, 45.

107. Set up a generating function and use it to find the number of ways in which eleven identical coins can be put in three distinct envelopes (labeled A, B, C) envelopes A and B have the same number of coins in them.
Ans: $(1 + x^2 + x^4 + x^6 + x^8 + x^{10})(1 + x + x^2 + x^3 + ...)$, 6.

108. Set up a generating function and use it to find the number of ways in which nine identical blocks can be given to four children if each child gets at least one block.
Ans: $(x + x^2 + x^3 + ...)^4$, 56.

109. Set up a generating function and use it to find the number of ways in which nine identical blocks can be given to four children, if each child gets at least two blocks.
Ans: $(x^2 + x^3 + x^4 + ...)^4$, 4.

110. Set up a generating function and use it to find the number of ways in which nine identical blocks can be given to four children, if each child gets at most five blocks.
Ans: $(1 + x + x^2 + x^3 + x^4 + x^5)^4$, 140.

111. Set up a generating function and use it to find the number of ways in which nine identical blocks can be given to four children, if the oldest child gets three blocks.
Ans: $x^3(1 + x + x^2 + x^3 + ...)^3$, 28.

112. Set up a generating function and use it to find the number of ways in which nine identical blocks can be given to four children, if the oldest child gets at most three blocks.
Ans: $(1 + x + x^2 + x^3)(1 + x + x^2 + x^3 + ...)^3$, 164.

113. Set up a generating function and use it to find the number of ways in which nine identical blocks can be given to four children, if the oldest child gets either 2 or 3 blocks.
Ans: $(x^2 + x^3)(1 + x + x^2 + x^3 + ...)^3$, 64.

114. If $G(x)$ is the generating function for $a_0, a_1, a_2, a_3, ...$, describe in terms of $G(x)$ the generating function for $0,0,0,a_0,a_1,a_2,...$.
Ans: $x^3 G(x)$.

115. If $G(x)$ is the generating function for $a_0, a_1, a_2, a_3, ...$, describe in terms of $G(x)$ the generating function for $0,0,0,a_3,a_4,a_5,...$.
Ans: $G(x) - a_0 - a_1 x - a_2 x^2$.

116. If $G(x)$ is the generating function for $a_0, a_1, a_2, a_3, \ldots$, describe in terms of $G(x)$ the generating function for $a_3, a_4, a_5, a_6, \ldots$.

Ans: $\dfrac{1}{x^3}\left(G(x) - a_0 - a_1 x - a_2 x^2\right)$.

117. If $G(x)$ is the generating function for $a_0, a_1, a_2, a_3, \ldots$, describe in terms of $G(x)$ the generating function for $a_0, 0, a_1, 0, a_2, 0, a_3, 0, a_4, \ldots$.

Ans: $G(x^2)$.

118. If $G(x)$ is the generating function for $a_0, a_1, a_2, a_3, \ldots$, describe in terms of $G(x)$ the generating function for $a_0, 3a_1, 9a_2, 27a_3, 81a_4, \ldots$.

Ans: $G(3x)$.

119. If $G(x)$ is the generating function for $a_0, a_1, a_2, a_3, \ldots$, describe in terms of $G(x)$ the generating function for $a_0, 0, 0, a_1, 0, 0, a_2, 0, 0, a_3, \ldots$.

Ans: $G(x^3)$.

120. If $G(x)$ is the generating function for $a_0, a_1, a_2, a_3, \ldots$, describe in terms of $G(x)$ the generating function for $5, a_1, 0, a_3, a_4, a_5, \ldots$.

Ans: $G(x) - a_0 - a_2 x^2 + 5$.

121. Use generating functions to solve $a_n = 5a_{n-1} + 3$, $a_0 = 2$.

Ans: $a_n = -\frac{3}{4} + \frac{11}{4} \cdot 5^n$.

122. Use generating functions to solve $a_n = 7a_{n-1} - 10a_{n-2}$, $a_0 = 1$, $a_1 = 1$.

Ans: $a_n = -\frac{1}{3} \cdot 5^n + \frac{4}{3} \cdot 2^n$.

123. Use generating functions to solve $a_n = 3a_{n-1} + 2^n + 5$, $a_0 = 1$.

Ans: $a_n = \frac{11}{2} \cdot 3^n - 2^{n+1} - \frac{5}{2}$.

124. Find $|A_1 \cup A_2 \cup A_3 \cup A_4|$ if each set A_i has 100 elements, each intersection of two sets has 60 elements, each intersection of three sets has 20 elements, and there are 10 elements in all four sets.

Ans: 110.

125. Find $|A_1 \cup A_2 \cup A_3 \cup A_4|$ if each set A_i has 150 elements, each intersection of two sets has 80 elements, each intersection of three sets has 20 elements, and there are no elements in all four sets.

Ans: 200.

126. Find the number of terms in the formula for the number of elements in the union of four sets given by the principle of inclusion-exclusion.

Ans: 15.

127. Find the number of positive integers ≤ 1000 that are multiples of at least one of 3,5,11.
Ans: 515.

128. Find the number of positive integers ≤ 1000 that are multiples of at least one of 2,6,12.
Ans: 500.

129. Find the number of positive integers ≤ 1000 that are multiples of at least one of 3,4,12.
Ans: 542.

130. Suppose $|A| = |B| = |C| = 100, |A \cap B| = 60, |A \cap C| = 50, |B \cap C| = 40$, and $|A \cup B \cup C| = 175$. How many elements are in $A \cap B \cap C$?
Ans: 25.

131. How many positive integers not exceeding 1000 are not divisible by either 4 or 6?
Ans: 667.

132. A doughnut shop sells 20 kinds of doughnuts. You want to buy 30 doughnuts. How many possibilities are there if you want at most six of any one kind?
Ans: $\binom{49}{19} - \binom{20}{1}\binom{42}{19} + \binom{20}{2}\binom{35}{19} - \binom{20}{3}\binom{28}{19} + \binom{20}{4}\binom{21}{19}$.

133. A doughnut shop sells 20 kinds of doughnuts. You want to buy 30 doughnuts. How many possibilities are there if you want at most 12 of any one kind?
Ans: $\binom{49}{19} - \binom{20}{1}\binom{36}{19} + \binom{20}{2}\binom{23}{19}$.

134. A market sells ten kinds of soda. You want to buy 12 bottles. How many possibilities are there? if you want
(a) at least one of each kind
(b) at most seven bottles of any kind?
Ans: (a) $\binom{11}{2}$ (b) $\binom{21}{9} - \binom{10}{1}\binom{13}{9}$.

135. A market sells ten kinds of soda. You want to buy 12 bottles. How many possibilities are there? if you want at most three bottles of any kind?
Ans: $\binom{21}{9} - \binom{10}{1}\binom{17}{9} + \binom{10}{2}\binom{13}{9} - \binom{10}{3}\binom{9}{9}$.

136. Suppose you have 100 identical marbles and five jars (labeled A, B, C, D, E). In how many ways can you put the marbles in the jars if:
(a) each jar has at least six marbles in it?
(b) each jar has at most forty marbles in it?
Ans: (a) $\binom{74}{4}$. (b) $\binom{104}{4} - \binom{5}{1}\binom{63}{4} + \binom{5}{2}\binom{22}{4}$.

137. How many ways are there to choose five donuts if there are eight varieties and only the type of each donut matters?

Ans: $\binom{12}{7}$.

138. A market sells 40 kinds of candy bars. You want to buy 15 candy bars.
    (a) How many possibilities are there?
    (b) How many possibilities are there if you want at least three peanut butter bars and at least five almond bars?
    (c) How many possibilities are there if you want exactly three peanut butter bars and exactly five almond bars?
    (d) How many possibilities are there if you want at most four toffee bars and at most six mint bars?

Ans: (a) $\binom{54}{39}$. (b) $\binom{46}{39}$. (c) $\binom{44}{37}$. (d) $\binom{54}{39} - \binom{49}{39} - \binom{47}{39} + \binom{42}{39}$.

139. How many permutations of all 26 letters of the alphabet are there that contain at least one of the words DOG, BIG, OIL?

Ans: $24! \cdot 3$.

140. How many permutations of all 26 letters of the alphabet are there that contain at least one of the words CART, SHOW, LIKE?

Ans: $3 \cdot 23! - 3 \cdot 20! + 17!$.

141. How many permutations of all 26 letters of the alphabet are there that contain at least one of the words SWORD, PLANT, CARTS?

Ans: $3 \cdot 22! - 18!$.

142. How many permutations of all 26 letters of the alphabet are there that contain none of the words: SAVE, PLAY, SNOW?

Ans: $26! - 3 \cdot 23! + 20!$.

143. How many permutations of all 26 letters of the alphabet are there that contain at least one of the words: CAR, CARE, SCAR, SCARE?

Ans: $24!$.

144. How many permutations of the 26 letters of the alphabet are there that do not contain any of the following strings: LOP, SLOP, SLOPE, LOPE.

Ans: $26! - 24!$.

145. You have ten cards, numbered 1 through 10. In how many ways can you put the ten cards in a row so that card i is not in spot i, for $i = 1, 2, ..., 10$?

Ans: $D_{10} = 10! - \binom{10}{1} 9! + \binom{10}{2} 8! - \binom{10}{3} 7! + ... + \binom{10}{10} 0!$.

146. Suppose $|A| = 8$ and $|B| = 4$. Find the number of functions $f : A \to B$ that are onto B.

Ans: $4^8 - \binom{4}{1}3^8 + \binom{4}{2}2^8 - \binom{4}{3}1^8$.

147. An office manager has four employees and nine reports to be done. In how many ways can the reports be assigned to the employees so that each employee has at least one report to do.

Ans: $4^9 - \binom{4}{1}3^9 + \binom{4}{2}2^9 - \binom{4}{3}1^9$.

148. An office manager has five employees and 12 projects to be completed. In how many ways can the projects be assigned to the employees so that each employee works on at least one project.

Ans: $5^{12} - \binom{5}{1}4^{12} + \binom{5}{2}3^{12} - \binom{5}{3}2^{12} + \binom{5}{4}1^{12}$.

149. Find the number of ways to put eight different books in five boxes, if no box is allowed to be empty.

Ans: $5^8 - \binom{5}{1}4^8 + \binom{5}{2}3^8 - \binom{5}{3}2^8 + \binom{5}{4}1^8$.

150. Find the number of bit strings of length eight that contain a pair of consecutive 0s.

Ans: $a_n = a_{n-1} + a_{n-2} + 2^{n-2}$, $a_1 = 0$, $a_2 = 1$. Hence $a_8 = 201$.

151. Find the number of ways to climb a 12-step staircase, if you go up either one or three steps at a time.

Ans: $a_n = a_{n-1} + a_{n-3}$, $a_1 = a_2 = 1$, $a_3 = 2$. Hence $a_{12} = 60$.

152. Find the number of strings of 0s, 1s, and 2s of length six that have no consecutive 0s.

Ans: $a_n = 2a_{n-1} + 2a_{n-2}$, $a_1 = 3, a_2 = 8$. Hence, $a_6 = 448$.