Generating Functions

Ngày 17 tháng 11 năm 2012

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In this section we shall explore the interaction among polynomials, power series and counting.

Definition

The function $f(x) = \sum_{n=0}^{\infty} a_n x^n$ *is the* **genrating function** *of the sequence* a_n.

The funciton $f(x) = \sum_{k=0}^{\infty} \frac{a_k x^k}{k!}$ *is the* **exponential generating function** *of the sequence* a_k.

Examples:

The generating function of the sequence $1, 1, 1, \ldots$
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If $f(x) = \sum_{n=0}^{\infty} a_n x^n$, $g(x) = \sum_{n=0}^{\infty} b_n x^n$ then:

$$f(x)g(x) = \sum_{n=0}^{\infty} c_n x^n \text{ where } c_n = \sum_{k=0}^{n} a_k b_{n-k}$$

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5. But we can do better, Consider the function $g(x) = (\sum_{i=0}^{\infty} x^i)^4$.
6. Again, the coefficient of x^{27} in the Taylor expansion of this function is the answer.
7. We noticed that $\sum\limits_{i=0}^{\infty} x^i = \frac{1}{1-x}$.
8. So the answer will be the coefficient of x^{27} in the expansion of:
$$(1 - x)^{-4}$$

The generalized binomial coefficients

Recall: $(1+x)^n = \sum_{k=1}^{\infty} \binom{n}{k} x^k$.

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How can we use it for solving counting problems?

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2. *The coefficient of x^{70} in the product* $(1+x+\ldots+x^{30})(1+x+\ldots+x^{40})(1+x+\ldots+x^{50})$ *is the answer.*
3. *Note that:*

$$(1+x+\ldots+x^{30})(1+x+\ldots+x^{40})(1+x+\ldots+x^{50}) =$$

$$\frac{1-x^{31}}{1-x}\frac{1-x^{41}}{1-x}\frac{1-x^{51}}{1-x} = (1-x)^{-3}(1-x^{31})(1-x^{41})(1-x^{51})$$

.

Examples

All we need is to find the coefficient of x^{70} in:

$$\left(\sum_{i=0}^{\infty} \binom{-3}{i} x^i \right) (1 - x^{31} - x^{41} - x^{51} + \ldots)$$

which turns out to be 1061 once we understand the meaning of

$$\binom{-3}{i}$$

.

Drill

Use this technique to find the number of distinct solution to:

$$x_1 + x_2 + x_3 + x_4 = 85$$

$$10 \le x_1 \le 25, \; 15 \le x_2 \le 30, \; 10 \le x_3 \le 40, \; 15 \le x_4 \le 25.$$

The Generalized Binomial Theorem

Theorem (The generalized binomial theorem)

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For negative integers we get:

$$\binom{r}{i} = \frac{r(r-1)\dots(r-i+1)}{i!} = (-1)^i \binom{-r+i-1}{-r-1}$$

Drill

Show that:

$$\binom{\frac{1}{2}}{k} = \frac{(-1)^k}{4^k}\binom{2k}{k}$$

Derangements

Recall: an n-derangement is an n-permutation $\pi = a_1 a_2 \ldots a_n$ in which $\forall i: \ a_i \neq i$. If we denote the number of n-derangments by D_n then:

$$D_1 = 0, \ D_2 = 1 \ and \ D_{n+1} = n(D_n + D_{n-1}).$$

Let: $D(x) = \sum_{n=0}^{\infty} D_n \frac{x^n}{n!}$ (the exponential generating function for D_n).

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Or: $\frac{D_n}{n!} = \sum_{k=0}^{n} \frac{(-1)^k}{k!} \Longrightarrow D_n = n! \sum_{k=0}^{n} \frac{(-1)^k}{k!}$

Question

You need to calculate the product of n matrices $A_1 \times A_2 \times \ldots \times A_n$*. How do we parenthesize the expression to do it in the most economical way?*

Catalan Numbers

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Why does it matter?

Drill

Let $A[m, n]$ *denote an* $m \times n$ *matrix (m rows and n columns). For each possible multiplication of the following product calculate the number of multiplications of real numbers needed to calculate the product.*

$$A[10, 20]A[20, 40]A[40, 50]A[50, 10]$$

Catalan Numbers

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e. *For $k \geq 0,\ A_1 A_2 \ldots A_{n+1}$ can be parenthesized as: $[A_1 \ldots A_k][A_{k+1} \ldots A_{n+1}]$ so the number of ways to further parenthesize this product is $m_{k-1} m_{n-k}$.*

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$$Hence: \qquad m_{n+1} = \sum_{i=0}^{n} m_i \cdot m_{n-i}$$

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7. Since $2xA(x) = 0$ *when* $x = 0$ we have:
$$A(x) = \frac{1}{2x}(1 - \sqrt{1-4x}).$$

Catalan Numbers

Or:

Or: Substituting the initial condition $m_0 = A(0) = 0$ we get:

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$$(1-4x)^{\frac{1}{2}} = \sum_{k=0}^{\infty} \binom{1/2}{k} (-4)^k x^k = \sum_{k=0}^{\infty} \binom{2k}{k} x^k$$

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$\left(Using : \binom{1/2}{k} = (-1/4)^k \binom{2k}{k} \right)$.

m_n is the coefficient of x^n in the expansion of: $(1 - \sqrt{1 - 4x})/(1/2x)$

A simple calculation yields:

$$m_n = \frac{1}{n+1} \binom{2n}{n}$$

Lattice walks

Question

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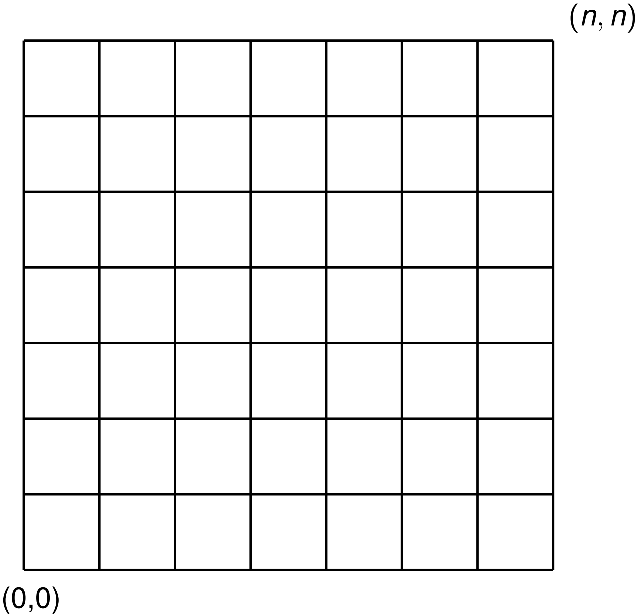
So the answer is:

$$\binom{2n}{n}$$

Question

The same question but this time your walk is restricted to stay below the diagonal.


(n, n)
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Answer

- *It may not look clear how to construct a solution, a recurrence relation, or just solve it.*
- *Every walk is a sequence $x_1, x_2, \ldots x_{2n}$ of moves where x_i is either move right or move up.*

Question

A minor change: We want to count the number of moves that stay below the diagonal.

Answer

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- *To stay below the diagonal, for each k the subsequence $x_1, x_2, \ldots x_k$ must have at least as many right-moves as up-moves.*

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- *Balanced parenthesis* $(()((()())),\ (: \rightarrow) : \uparrow$ *.*
  So the number of walks is the Catalan number m_{2n}*.*

Other counting problems can be solved by “mapping” them to solved problems.

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- n Persons line up to buy tickets to the theater. The cost of a ticket is 50,000 VND. Each person has a 50,000 VND or a 100,000 VND. The cashier opens the box office with no money. So if the first person has a 100,000 VND the line will get stuck as the cashier will not be able to give him change. In how many ways can n persons arrange the line so all of them will be able to buy tickets with no delays?

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- We need to assume that at least $\lceil \frac{n}{2} \rceil$ have a 50,000 VND note.