Recurrence Relations

Ngày 17 tháng 11 năm 2012

Recursive Problem Solving

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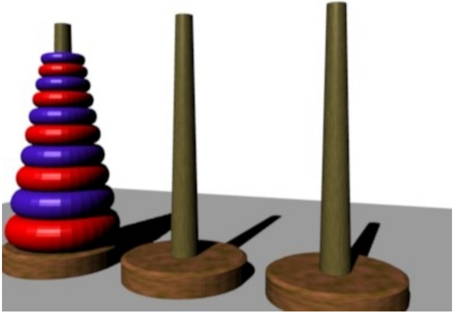
If we denote the number of bacteria at second number k by b_k then we have: $b_{k+1} = 2b_k,\ b_1 = 1$.

This is a recurrence relation.

The Towers of Hanoi

Another example of a problem that lends itself to a recurrence relation is a famous puzzle: **The towers of Hanoi**



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Clearly, before we move the large disk from the left to the right, all but the bottom disk, have to be on the middle tower. So if we denote the smallest number of moves by h_n then we have:

$$h_{n+1} = 2h_n + 1$$

A simple technique for solving recurrence relation is called *telescoping*.

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A simple technique for solving recurrence relation is called *telescoping*.

Start from the first term and sequntially produce the next terms until a clear pattern emerges. If you want to be mathematically rigoruous you may use induction.

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$$h_1 = 1,\ h_2 = 3,\ h_3 = 7,\ h_4 = 15, \ldots\ h_n = 2^n - 1$$

Proof by induction:

1. $h_1 = 1 = 2^1 - 1$

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5. *Solve:* $a_n = \frac{1}{1+a_{n-1}},\ a_1 = 1.$
6. *Telescoping yields:* $1, \frac{1}{2}, \frac{2}{3}, \frac{3}{5}, \frac{5}{8}, \frac{8}{13}$

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Recurrence Relations Terminology

Definition

A recurrence relation for a sequence a_n is a relation of the form $a_{n+1} = f(a_1, a_2, \ldots, a_n)$.

We do not expect to have a useful method to solve all recurrence relations. This definition actually applies to any sequence! We shall break down the functions for which we do have effective methods to “solve” the recurrence relation. By solving we mean obtaining an explicit expression of the form $a_n = g(n)$. To accomplish this we need some terminology.

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A recurrence relation is **linear** *if:*

$$f(a_1, a_2, \ldots, a_n) = \sum_{i=1}^{n} h_i \cdot a_i + h(n) \text{ Where } h(n) \text{ is a function of } n.$$

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4. $a_n = a_{n-1} + 2a_{n-2} + 4a_{n-5} + 2^n$ is a non-homogeneous, linear recurrence relation with constant coefficients of order 5.

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3. This is a linear, homogeneous recurrence relation with constant coefficients, but not of finite order.

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If $f(n)$ and $g(n)$ are solutions to a non homgeneous recurrence relation then $f(n) - g(n)$ is a solution to the associated homogeneous recurrence relation.

Remark

This means that in order to solve a non homogeneous linear recurrence relation all we need to do is find the general solution $g(n)$ to the homogeneous part and a particular solution $p(n)$ to the non homogenesous equation.

The general solution will be: $g(n) + p(n)$.

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5. *Solving for c and d we get:* $a_n = \alpha 2^n - 3n - 5$

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Let $a_n = b \cdot a_{n-1} + c \cdot a_{n-2} + g(n), \quad a_1 = \alpha, \; a_2 = \beta.$
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Corollary

Any solution that satisfies the recurrence relation and initial conditions is THE ONLY solution.

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3. *Then the general solution of this recurrence relation is* $a_n = \alpha r_1^n + \beta r_2^n$.

Definition

Let $a_n = ba_{n-1} + ca_{n-2}$.
The quadratic equation $x^2 - bx - c = 0$ *is called the* **characteritic equation** *of the recurrence relation.*

Theorem (Solving Linear Homogeneous RR with Constant Coefficients)

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4. *If* $r_1 = r_2$ *then the general solution is* $a_n = \alpha r^n + \beta n r^n$

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4. Thus $\alpha r_1^n + \beta r_2^n$ solves the recurrence relation.
5. As previously proved, $r^n = br^{n-1} + cr^{n-2}$. Taking the derivative we get: $nr^{n-1} = b(n-1)r^{n-2} + c(n-2)r^{n-3}$ and if we multiply both sides by r we get: $nr^n = b(n-1)r^{n-1} + c(n-2)r^{n-2}$

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1. Let $a_0 = m,\ a_1 = k$. We need to show that we can choose α *and* β so that $\alpha r_1^0 + \beta r_2^0 = m$ *and* $\alpha r_1 + \beta r_2 = k$.

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3. In the second case we have: $\alpha = m$ *and* $\alpha + \beta = k$ which obviously has a solution.

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    - Solve for c: $c = 1$
    - General solution: $a_n = \alpha 3^n + n \cdot 3^n$

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   - Substitute and solve for c, d we find that $\frac{1}{3}n^3 + n^2$ is a particular solution.
   - So the general solution is: $a_n = \alpha + \beta n + n^2 + \frac{1}{3}n^3$.