Discrete Optimization

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1 Permutations

This short section will be devoted to a short review of permutations. It is a topic we hope you are familiar with. We shall start with the basic definitions, notation, fundamental properties, examples and in a separate pdf files a sample of questions, hints and answers.

Definition 1. *A bijection on a set A is called a* **permutation**

If $|A| = k$ then a permutation on A is a k-permutation. In most cases, the word permutation is used for permutations on finite sets.

The usual notation for a permutation is:

$$S = \begin{pmatrix} 1 & 2 & ,\dots, & k \\ a_1 & a_2 & ,\dots, & a_k \end{pmatrix} \tag{1}$$

An alternative simpler notation we shall use is $a_1 a_2 a_3 \dots a_k$.

Definition 2. *The product of two permutations $\pi_1 \times \pi_2$ is the composition of the two bijections definig them.*

Corolary 1. *The set of n-permutations on a set of cardinality n forms a group, the* symmetric group *S_n.*

Example 1. •

- *The permutation: $\pi = 1\ 3\ 5\ 4\ 2\ 6\ 8\ 10\ 9\ 7$ is a 10-permutation.*
- $\pi(1) = 1,\ \pi(5) = 2,\ \pi(7) = 8$

Definition 3.

1. *Let π be a k-permutation. A cycle is a subsequence $a_1\ a_2\ \dots\ a_m$ of π such that $\pi(a_i) = a_{i+1},\ i < m,$ and $\pi(a_m) = a_1$..*
2. *We denote the cyclic permutaions by $(a_1, a_2, \dots, a_m)$.*

3. *When it is clear from the context that we are dealing with n-permutations when we write* $(a_1, a_2, \ldots, a_m)$. *we mean an n-permutation* μ *such that* $\mu(j) = j$ *if* $j \notin \{a_1, a_2, \ldots, a_m\}$

Example 2. *The 10-permutation* π *in example 1. Contains the cycels* (1), $(2, 3, 5)$, (4) *etc.*

Exercise 1. •

- *Find all cycles in the permutation* π *in example 1.*
- *Check that they are pairwise disjoint.*
- *Check that* π *is the product of these cycles.*

Definition 4. *A cycle* (a, b) *is called a* **transposition**.

Note that if π is a transposition then π^2 is the identity permutation.

Remark 1. •

- *Every permutation is a unique product of disjoint cycles.*
- *Every cycle is a product of transpositions.*
- *Every permutation is a product of transposition.*
- *While the representation of permutations as a product of transpositions is not unique the parity is invariant. This means that all representations of a permutation as a product of transpositions are either all a product of an even number of transpositions, or and odd number of transpositions.*

Definition 5. •

- *An n-permutation that is a product of an even number of transpositions is called an even permutation.*
- *the set of all even n-permutations is a group denotde by* A_n.
- $|S_n| = n!,\ |A_n| = \frac{n!}{2}$.

Exercise 2. •

- *Is the permutation in example 1 odd or even?*
- *Construct a 12-permutation which is the product of 3 disjoint 4-cycles.*
- *Determine whether your permutation is odd or even.*
- *Find its inverse.*
- *Is the cycle* $(1, 2, \ldots, n)$ *odd or even?*

We conclude this brief review of permutations by introducing an order on S_n.

Definition 6. *Given two n-permutations* $\pi = a_1\ a_2\ \ldots a_n$ *and* $\mu = b_1\ b_2\ \ldots b_n$ *we say that* $\pi \prec \mu$ *if* $a_i = b_i, \quad for\ i < j$ *and* $a_j < b_j$.

Example 3. 7 3 4 8 1 5 6 2 $\prec$ 7 3 4 8 2 1 6 5

This is a total order on the set of n-permutations. It is called the *l*exicographic order.

Exercise 3. *Can you find a permutation* α *between* π *and* μ*?*