Do all infinite sets have the same size?

Ngày 10 tháng 9 năm 2012

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Theorem (5)

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[Sketch of a proof for Theorem 1]

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Indeed given any function $f : A \to P(A)$ let
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Fill in the details.

Conclusion: since there is an injection $g : A \to P(A)$ and there is no onto function $f : A \to P(A)$ we conclude that $\mid A \mid < \mid P(A) \mid$.

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Remark

This proof technique is called the Diagonal Method. It is used on many occaisons. For instance Theorem 1 is an abstract form of this method.

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There are functions $f : N \to \{0,1\}$ (decision problems) that are not programmable.

Each program that implements a decision problem is stored in memory as a finite binary sequence. There are only countably many finite binary sequences. Hence there are non computable functions.

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Verify this assertion.

In Set Theory this is known as Bernstein's Lemma.

Surprise “Squaring the Circle”

Remark

There is a surprising consequence of this famous lemma. If you take two sets of points A and B in the plane, and if each set contains a disk, then each set can be disected into two sets A_1, A_2, B_1, B_2 *such that* A_i *and* B_i *are similar.*

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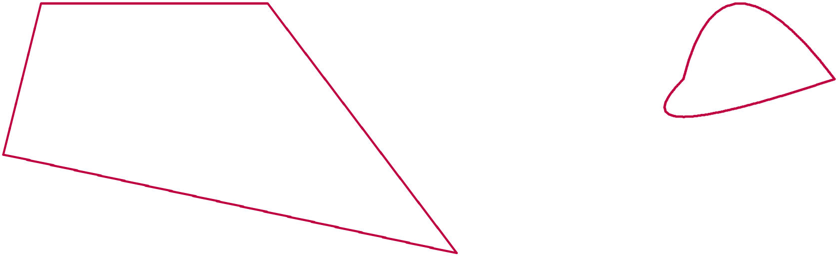

For example: these two sets can be disected into a pair of similar sets!