Do all infinite sets have the same size?

Ngày 8 tháng 9 năm 2012

FUNCTIONS, Preface

Functions are the muscles and blood of mathematics, the sciences and many other areas. This section may change drsatically your current notion of a **function**. One of our goals in introducing this notion here is to be able to answer some "simple" questions on sets: like how "large" can a set be? Given two sets, can we say which one is larger?

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4. Functions: a type of relation.

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Notation: $f : A \rightarrow B$.

Alternatively, $f \subset A \times B$ such that
$((a, b) \in f) \wedge ((a, c) \in f) \rightarrow b = c$.

In other words, a function $f : A \rightarrow B$ is a “restricted” binary relation between A and B.
Common notation: $f(a) = b$ b is the image of a under the function f.

Question

Which of the relations in our sample of 8 relations is a function?

Examples

[<+-|alert@+>]1. **f** assigns to a bit string the number of 1's in the string.Domain: $\{b \mid \text{All bit strings}\}$ Range $= \{0, 1, 2, \ldots\} = N$.**f**(0110101) = 4.

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In this section we shall develop the tools that will enable us to compare sets.

We will prove that there are “unlimited” sizes of sets and that there are many *non computable functions*.

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The function $f(n) = 2n$ is a bijection between the integers and the even integers.

This means that there is a bijection between a set and "half" its size!

The inverse function

We need a few more definitions to be ready for our goal.

Definition

A set B is finite if there is a bijection between B and N_k.
$(N_k = \{1, 2, \ldots, k\})$

If $f : A \to B$ is a bijection then we can define a new function $f^{-1} : B \to A$, the inverse of f, as follows: to find how f^{-1} maps the element $b \in$B find the unique $a \in A$ such that: $f(a) = b$ and define $f^{-1}(b) = a$.

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Example

$f(x) = 3x + 1,\ x \in R$.
$f^{-1}(x) =?$

Definition

Let $g : A \to$B and $f : B \to C$. The **composition** of the functions f and g, denoted by $f \circ g$ is a function $f \circ g : A \to C$ defined by $f \circ g(a) = f(g(a))$.

Observation: If $f : A \to B$ and $g : B \to C$ are bijections then $g \circ f : A \to C$ and $f^{-1} \circ g^{-1} : C \to A$ are also a bijections..

If f is a function on the set A, then $f \circ I(a) = I \circ f(a) = f(a)$.

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The function $f : A \rightarrow A$ defined by $f(a) = a \, \forall a \in A$ is called the **Identity** function. We denote it by I.

If f is a function on the set A, then $f \circ I(a) = I \circ f(a) = f(a)$.

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$f \circ g(x)$ and $g \circ f(x)$ can be distinct functions, or the composition is not commutative.

The bijections on a set A form a group.

Theorem

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3. *In particular, if $|A| \geq |B| \wedge |B| \geq |A|$ does it imply that $|A| = |B|$?*

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Countable sets play a central role in discrete mathematics.

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There are other equivalent forms of the principle of mathematical induction:

1. $1 \in A, \; (\forall k < n, \; k \in A \rightarrow n \in A)$ then $A = Z^+$.

2. *If* $(\exists a_n \in A, \; a_n \rightarrow \infty) \rightarrow (a_n - 1) \in A$ then $A = Z^+$.

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Theorem (4)

If $\mid A \mid \leq \mid B \mid$ *and* $\mid B \mid \leq \mid A \mid$ *then* $\mid A \mid = \mid B \mid$

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Indeed given any function $f : A \to P(A)$. Let $S = \{a \in A \mid a \notin f(a)\}$.
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Whether $s \in f(s)$ *or* $s \notin f(s)$ we reach a contradicion.
Fill in the details.
Conclusion: since there is an injection $g : A \to P(A)$ and there is no onto function $f : A \to P(A)$ we conclude that $\mid A \mid < \mid P(A) \mid$.

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Let $y = 0.y_1y_2\ldots y_n\ldots$ be defined as follows:

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Fill in the details, that is prove that $y \notin A$.

Proofs

[Sketch of a proof for theorem 2] For every countable set $A \subset \{x \mid 0 < x < 1,\ x \in R\} = \mathbb{U}$ we shall find a real number $y \notin A$. Let $\{x_1, x_2, \ldots, x_n, \ldots\}$ be a countable subset of $\mathbb{U}$. Let $x_n = 0.x_{n,1}x_{n,2}\ldots x_{n,n}x_{n,n+1}\ldots$ be the decimal expansion of x_n.

Let $y = 0.y_1y_2\ldots y_n\ldots$ be defined as follows:

Let $y_n = x_{n,n} + 5 \pmod{10}$. We want to make sure that $\forall n,\ y_n \neq x_{n,n}$.

Fill in the details, that is prove that $y \notin A$.

Remark

This proof technique is called the Diagonal Method. It is used on many occaisons. For instance Theorem 1 is an abstract form of this method.

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Here we go again.
[Theorem 3, proof sketch] It is enough to show that there is a bijection between the set of functions: $\{f : N \to \{0, 1\}\}$ and $P(N)$.

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[of the corollary] Each program that implements a decision problem is stored in memory as a finite binary sequence. There are only countably many finite binary sequences. Hence there are non computable functions.

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Verify: Each chain is one of the following four types:

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Verify this assertion.

In Set Theory this is known as bernstein's Lemma.

Surprise

Remark

There is a surprising consequence of this famous lemma. If you take two sets of points A and B in the plane, and if each set contains a disk, then each set can be disected into two sets A_1, A_2, B_1, B_2 *such that* A_i *and* B_i *are similar.*

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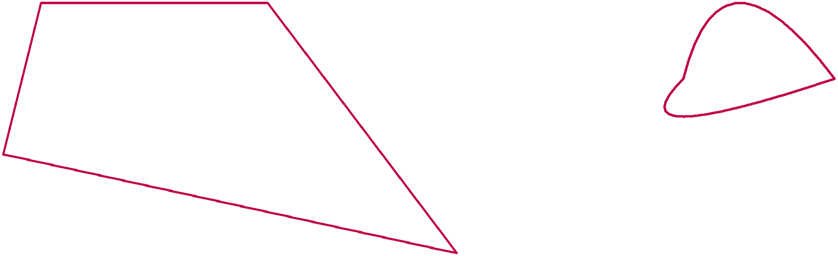

For example: these two sets can be disected into a pair of similar sets!