Discrete Mathematics
and its Applications

Ngày 8 tháng 9 năm 2012

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7. *Let G be a group and H a subgroup of G.* $\mathbb{R}_8 = \{(r,s) \mid r \cdot s^{-1} \in H\}$ *is a relation on G.*

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We say that the relation $\mathbb{R}$ partitions A into equivalence classes.

Relations

Recall: relations are sets. So we can create new relations from given ones by using set operations. We can take unions, intersections, other set operations and create new relations.

Example (Compositions)

Given a binary relation $\mathbb{R}_1$ between A and B and a second binary relation $\mathbb{R}_2$ between B and C we can define a new relation $\mathbb{R}_3$ between A and C, the **composition** *of $\mathbb{R}_1$ and $\mathbb{R}_2$ as follows:*

$$\mathbb{R}_1 \circ \mathbb{R}_2 = \{(a, c) \mid \exists b \in B \textit{ such that } (a, b) \in \mathbb{R}_1 \textit{ and } (b, c) \in \mathbb{R}_2\}$$

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Can there be more than one "smallest"?

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Remark

The transitive closure is an operation very frequently executed in many applications. Facebook is one of them. Warshall's algorithm effciently produces the transitive closure of a relation.

Warshall's transitive closure algorithm

Let $\mathbb{R}$ be a relation on the finite set $A = \{a_1, a_2, \ldots, a_n\}$.
A list $a_{i_1}, a_{i_2}, \ldots a_{i_k}$ is a **path** if $(a_{i_j}, a_{i_{j+1}}) \in \mathbb{R}$.

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1. Initialize: $\mathbb{R}_1 = \mathbb{R}$

1: $\forall$ pairs (i,j) add (a_i, a_j) to $\mathbb{R}_1$ if $(a_i, a_1) \wedge (a_1, a_j) \in \mathbb{R}_1$.

2: $\forall$ pairs $(i.j)$ add (a_i, a_j) to $\mathbb{R}_1$ if there is a path a_i, a_1, a_j or a path a_i, a_2, a_j or a_i, x, y, a_j, $\{x, y\} = \{a_1, a_2\}$ in $\mathbb{R}_1$.

t: $\forall$ pairs (i,j) add (a_i, a_j) to $\mathbb{R}_1$ if there is a path $a_i, x_1, \ldots, x_m, a_j$ in $\mathbb{R}_1$.

Example

Let us find the transitive closure of the relation:
$\{(1,2), (2,3), (3,4), (2,1), (1,4)\}$