Discrete Mathematics Lecture-16 Communication Security

November 17, 2012



Question

Mr. Nguyen sells expensive jewelry. He has an interesting idea for a business model. Each customer will have access to boxes with a combination lock. Once a person grabs a box he can set his own private combination lock. An open box can be closed by anyone, but only the owner knows the combination and can open it. The content of any open box sent between persons will be stolen.

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How can we accomplish this?

This is exactly how business transactions are being conducted on the Internet today, except that the boxes are virtual boxes. Closing a box is accomplished by encrypting the message. So while the message is traveling on the Internet, being exposed to hackers and others, it is encrypted using a "key". Only the owner of the key knows how to open the box and retrieve its content.

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- Messages can be sent to "Bob" so only Bob will be able to understand.
- Transactions can be "signed."

Discussion

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• Decryption: the receiver calculates $S^d \mod k$ and retrieves M where $d = e^{(-1)} \mod (p-1)(q-1)$.

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We shall devote the rest of our time to take a quick glimpse at factoring.

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After all, all they need to do is calculate $d = e^{(-1)} \mod (p-1)(q-1)$ and in order to do it they just need to factor k.

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To understand it we need to study some very mathematically interesting topics in modular arithmetic.

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- A polynomial p(x) of degree k over GF(q) has at most k roots.



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Answer

Unfortunately not. There are numbers for which the chances for finding an integer a < n such that $a^{(n-1)} \mod n \neq 1$ are very slim. For instance if n = (6k+1)(12k+1)(18k+1) and

(6k + 1), (12k + 1) and (18k + 1) are prime, then if acd(a, n) = 1 $a^{n-1} \mod n = 1$.

Let N be an integer. By Fermat's theorem if N is prime then $a^{N-1} \mod N = 1$. This calculation can be executed very fast on integers with a few thousand digits. This means that if for some 1 < a < N-1; $a^{N-1} \mod N \neq = 1$ then N is definitely not a prime number.

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But what can we conclude if $a^{N-1} \mod N = 1$?

Answer: NOTHING! N may be prime and it may be composite! At best, we can try another integer a.

Example

As we noted in our drill, $k^{1728} \mod 1729 = 1$ for all k, gcd(k, 1729) = 1. Our chances to randomly select k such that gcd(k, 1729) > 1 are very slim.

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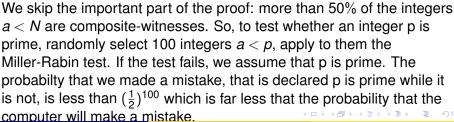
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- But $3^{3^3} = 664$ proving that 1729 is composite.
- Drill: Find a witness that will prove that 413138881 is composite.

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- Not really. There are some very sofisticated attacks on "weak" primes.

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Half the positive integres mod a prime number p are quadratic residues. While finding their square roots is not difficult it is a bit trickier than finding the square root of an integer.

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Note: $n = \alpha^m$ where α is a primitive number mod p. n is a quadratic residue mod p if and only if m = 2k.

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- This can be accomplished as follows:

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- For an example see the SAGE sample in the supplements folder.



See the file factoring.pdf