Discrete Mathematics
Lecture-16 Communication Security

November 17, 2012

Security Transactions

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How can we accomplish this?

Discussion

This is exactly how business transactions are being conducted on the Internet today, except that the boxes are virtual boxes. Closing a box is accomplished by encrypting the message. So while the message is traveling on the Internet, being exposed to hackers and others, it is encrypted using a "key". Only the owner of the key knows how to open the box and retrieve its content.

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- *Transactions can be "signed."*

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In 1976 Rivest, Shamir and Adelman proposed the public key cryptosystem: **RSA**.

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- Decryption: the receiver calculates $S^d \bmod k$ and retrieves M where $d = e^{(-1)} \bmod (p-1)(q-1)$.

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We shall devote the rest of our time to take a quick glimpse at factoring.

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After all, all they need to do is calculate $d = e^{(-1)} \bmod (p-1)(q-1)$ and in order to do it they just need to factor k.

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To understand it we need to study some very mathematically interesting topics in modular arithmetic.

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- A polynomial $p(x)$ of degree k over $GF(q)$ has at most k roots.

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Can Fermat’s theorem be used for testing primality?

Answer

Unfortunately not. There are numbers for which the chances for finding an integer $a < n$ *such that* $a^{(n-1)} \bmod n \neq 1$ *are very slim.*
For instance if $n = (6k+1)(12k+1)(18k+1)$ *and* $(6k+1), (12k+1)$ *and* $(18k+1)$ *are prime, then if* $gcd(a, n) = 1$ $a^{n-1} \bmod n = 1$.

Primality testing

Let N be an integer. By Fermat's theorem if N is prime then $a^{N-1} \bmod N = 1$. This calculation can be executed very fast on integers with a few thousand digits. This means that if for some $1 < a < N-1;\ a^{N-1} \bmod N \neq= 1$ then N is definitely not a prime number.

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But what can we conclude if $a^{N-1} \bmod N = 1$?

Answer: NOTHING! N may be prime and it may be composite! At best, we can try another integer a.

Example

As we noted in our drill, $k^{1728} \bmod 1729 = 1$ *for all* $k,\ gcd(k, 1729) = 1$. *Our chances to randomly select k such that* $gcd(k, 1729) > 1$ *are very slim.*

The Miller-Rabin Primality Test

Comment

Positive integers N for which $a^{N-1} \bmod N = 1$ $\forall a$ *such that* $gcd(a, N) = 1$ *are called* **Carmichael numbers**.

There are infinitely many Carmichael numbers.

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We skip the important part of the proof: more than 50% of the integers $a < N$ are composite-witnesses. So, to test whether an integer p is prime, randomly select 100 integers $a < p$, apply to them the Miller-Rabin test. If the test fails, we assume that p is prime. The probabilty that we made a mistake, that is declared p is prime while it is not, is less than $(\frac{1}{2})^{100}$ which is far less that the probability that the computer will make a mistake.

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- *But* $3^{3^3} = 664$ *proving that* 1729 *is composite.*
- *Drill: Find a witness that will prove that* 413138881 *is composite.*

Factoring large integers

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- Not really. There are some very sofisticated attacks on “weak” primes.

Square roots and factoring

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Half the positive integres mod a prime number p are quadratic residues. While finding their square roots is not difficult it is a bit trickier than finding the square root of an integer.

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- *This can be accomplished as follows:*

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- For an example see the SAGE sample in the supplements folder.

$\sqrt{n} \bmod p \cdot q$

See the file factoring.pdf