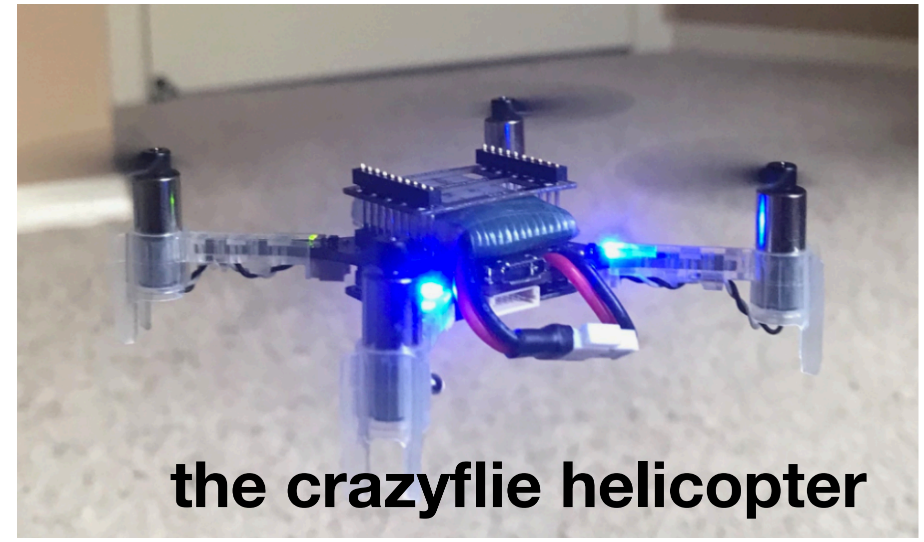
control and dynamics

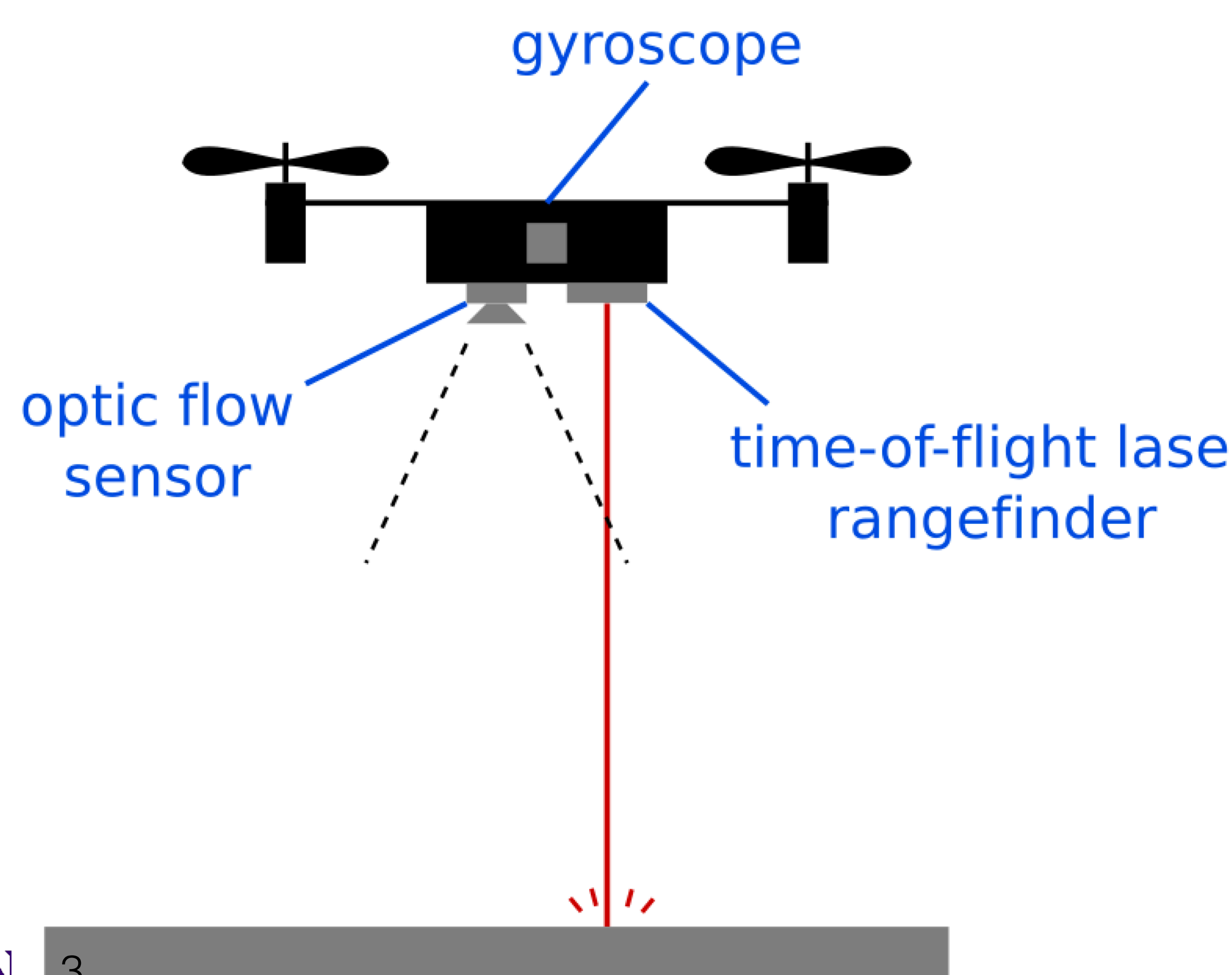
Prof. Sawyer B. Fuller

ME 586: Biology-inspired robotics

Project-based portion of this course

- you will work with the crazyflie helicopter as part of your homework problem sets
  - learning objectives:
    - learn basics of robotics and drone control
    - experience implementing bio-inspired control algorithm
- Crazyflie specs:
  - ~30 g, ~4 minute flight time
  - communicates in real-time over bluetooth to laptop
  - sensor suite gives information needed to stabilize and control flight
  - open-source control software





three ideas inspired by biology for how to improve robotics

(the themes of this course)

1. adaptation through evolution and learning

😀 fundamental engineering processes used by biology
☹️ "curse of dimensionality"

2. mechanical intelligence
   - the use of mechanics to reduce or eliminate the need for feedback control
3. parsimony
   - simple and efficient solutions

😀 "shortcut": look directly to biology for inspiration, combine with engineering knowledge

ME586 homework and projects emphasize these. We will show that the optimal control formulation we use for flight stability is also the basis for robot learning.

crazyflie in operation performing odor source localization

Odor Localization

Anderson, Sullivan, Horiuchi, Fuller, & Daniel, *Bioinspiration & Biomimetics* 2020

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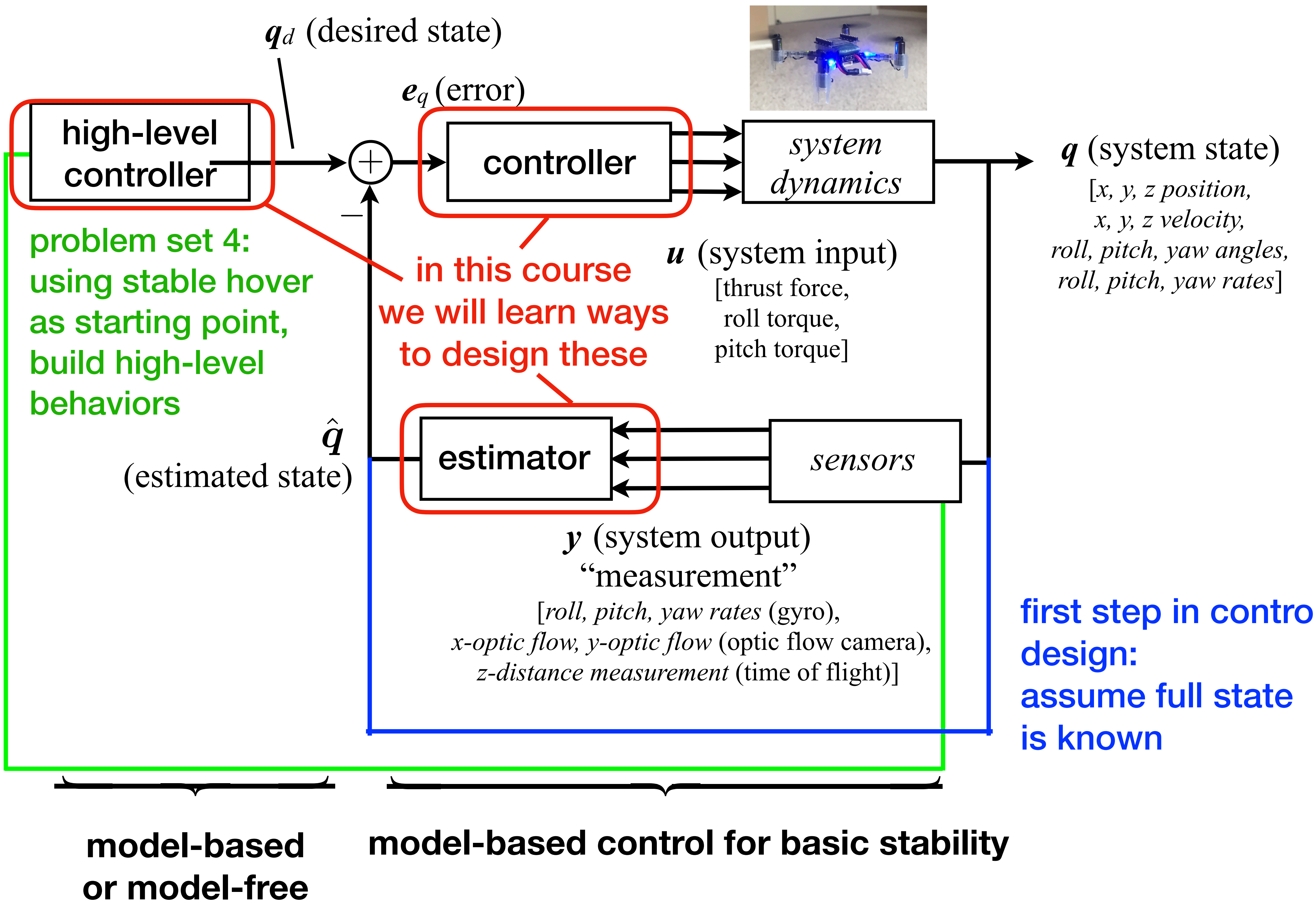
the controller we will learn



basics: actuation for hovering

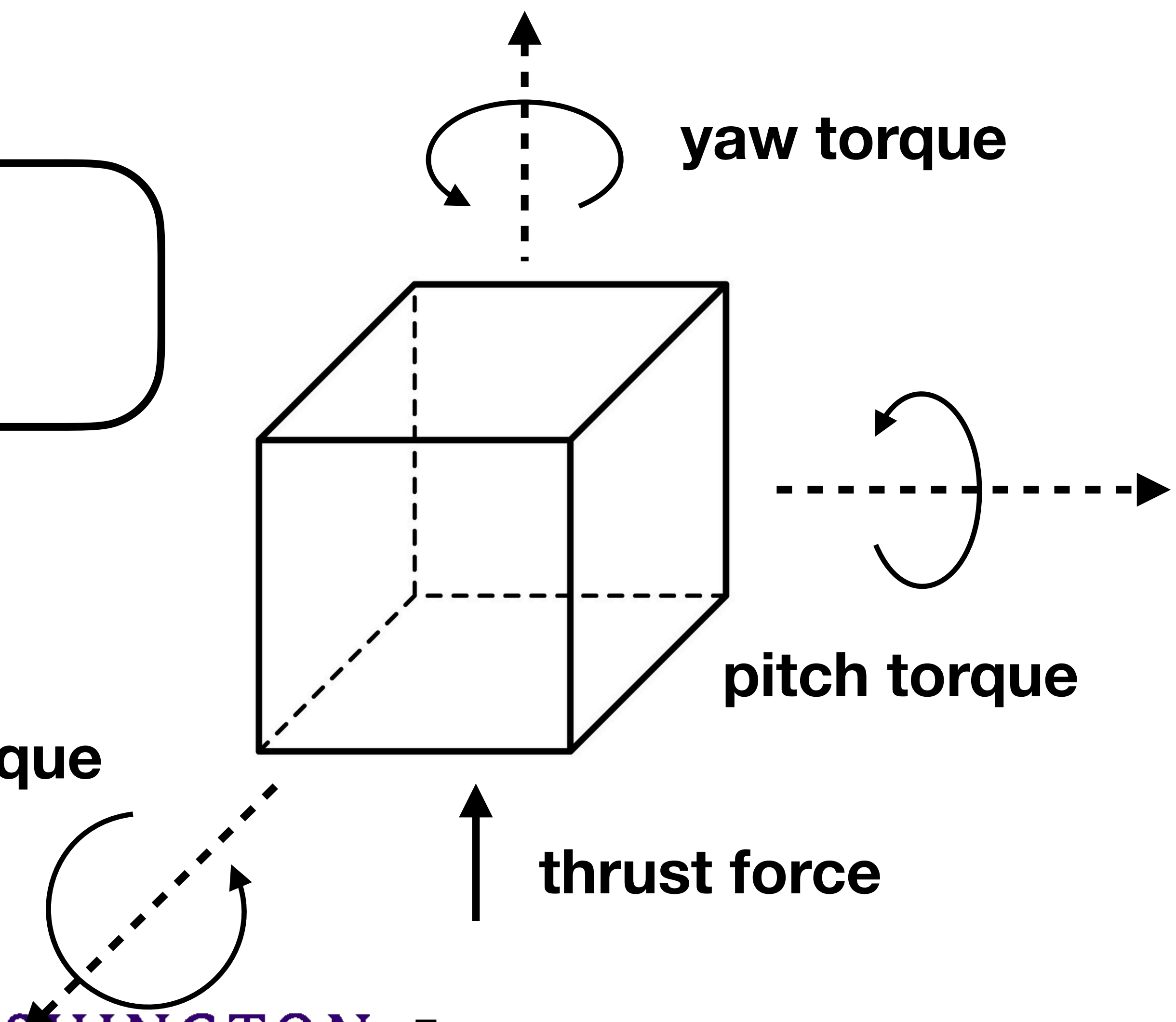
honeybee

single-rotor helicopter

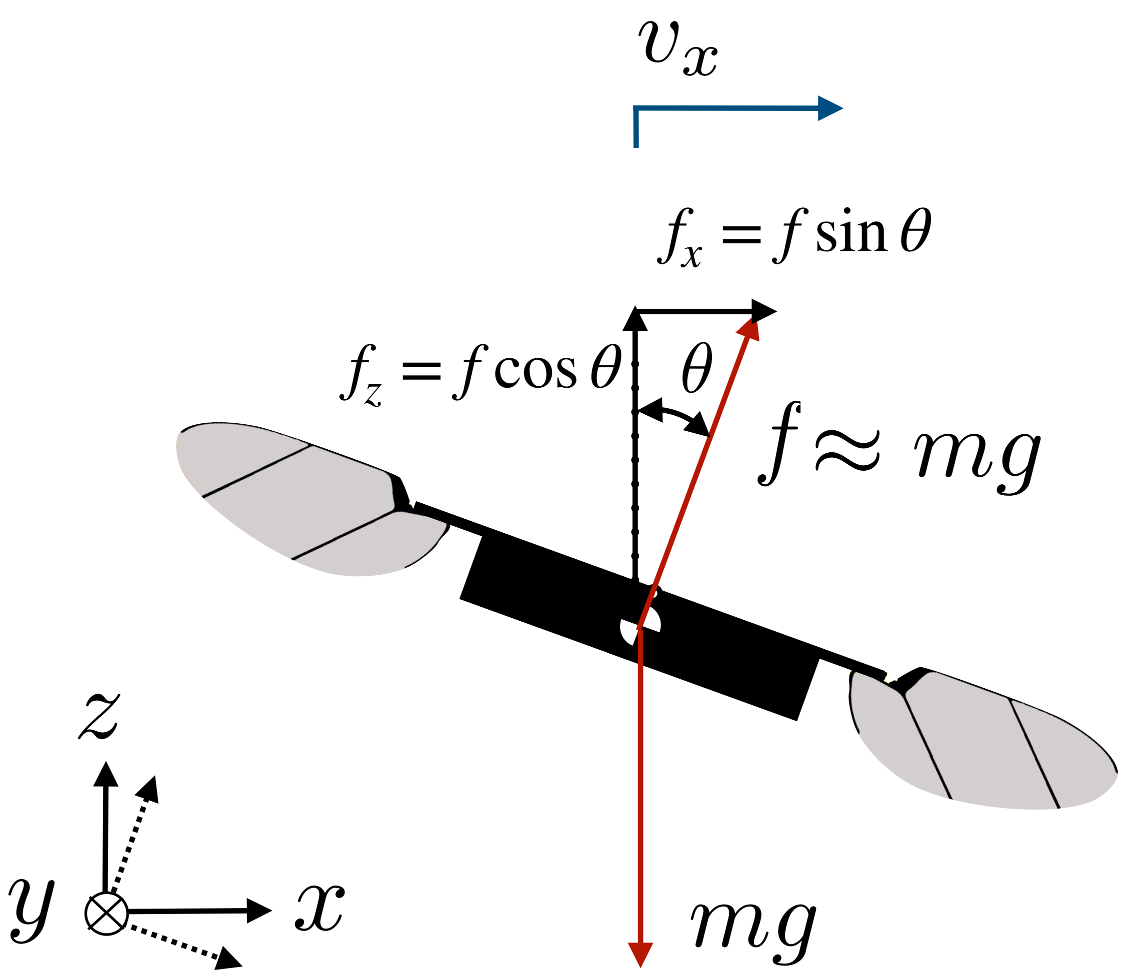
robot flies e.g.
UW Robofly

four-rotor aircraft
“quad-rotors”

**typically, lateral thrust
is not directly actuated**



If you can tilt, how do you move laterally?



lateral acceleration

$$m\dot{v}_x = f_x = f \sin\theta \approx mg \sin\theta$$

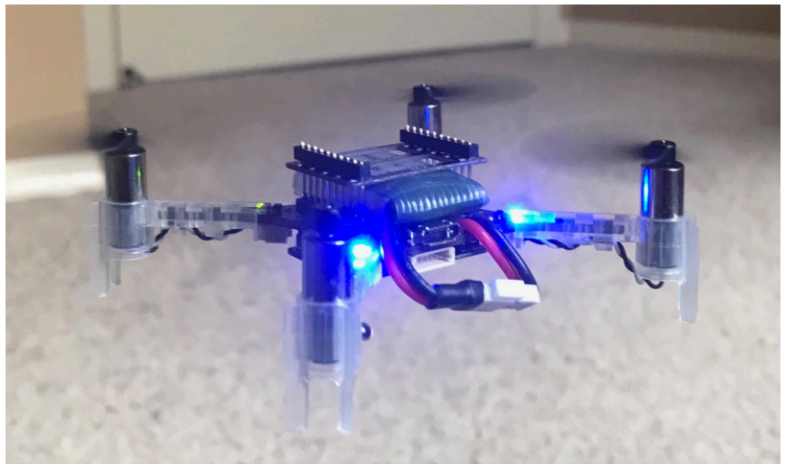
$$\Rightarrow \dot{v}_x = g \sin\theta$$

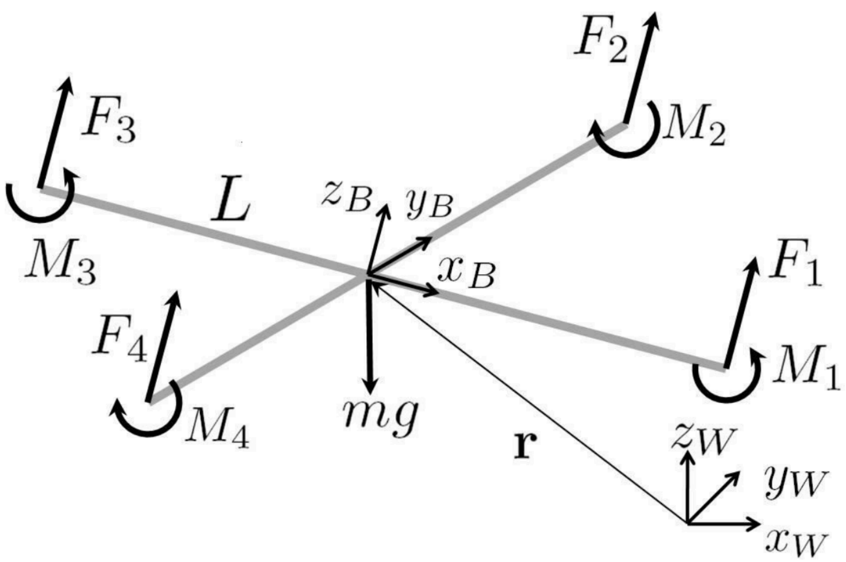
$$\approx g\theta \quad \textbf{for small } \theta$$

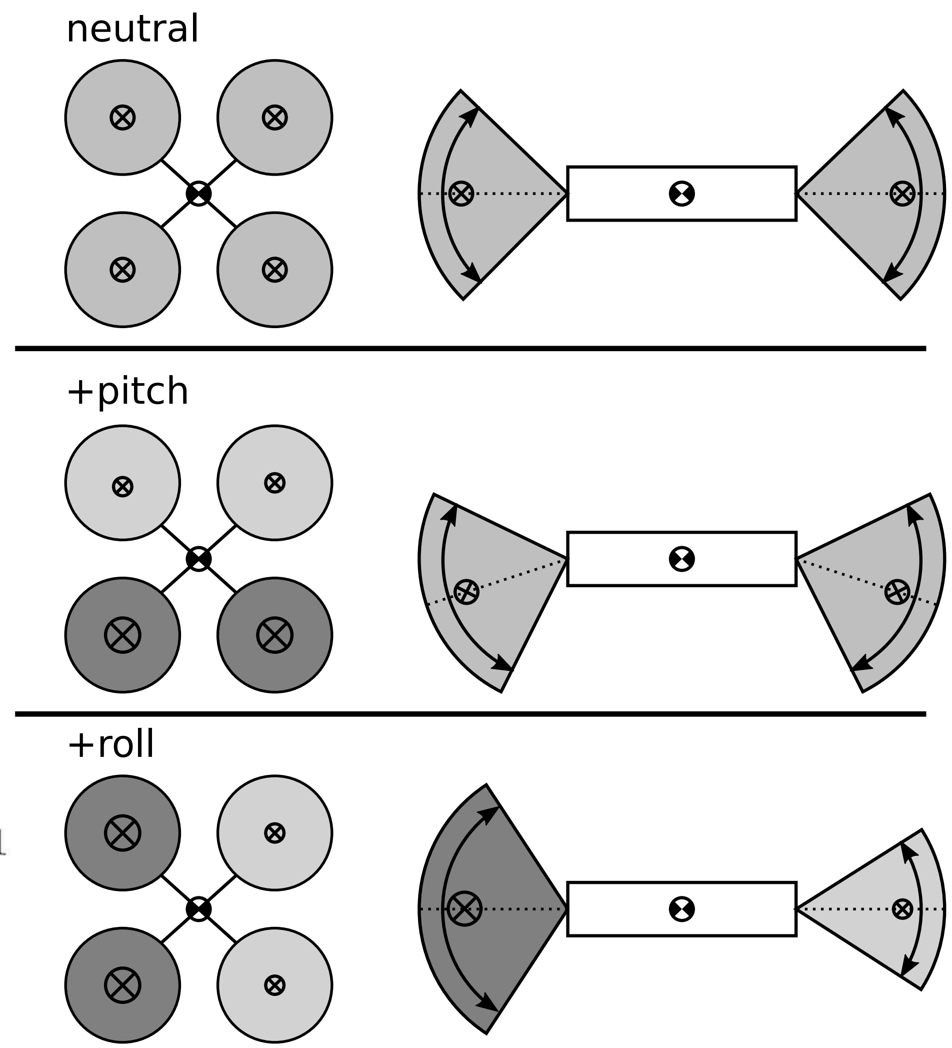
- "helicopter-like" lateral control

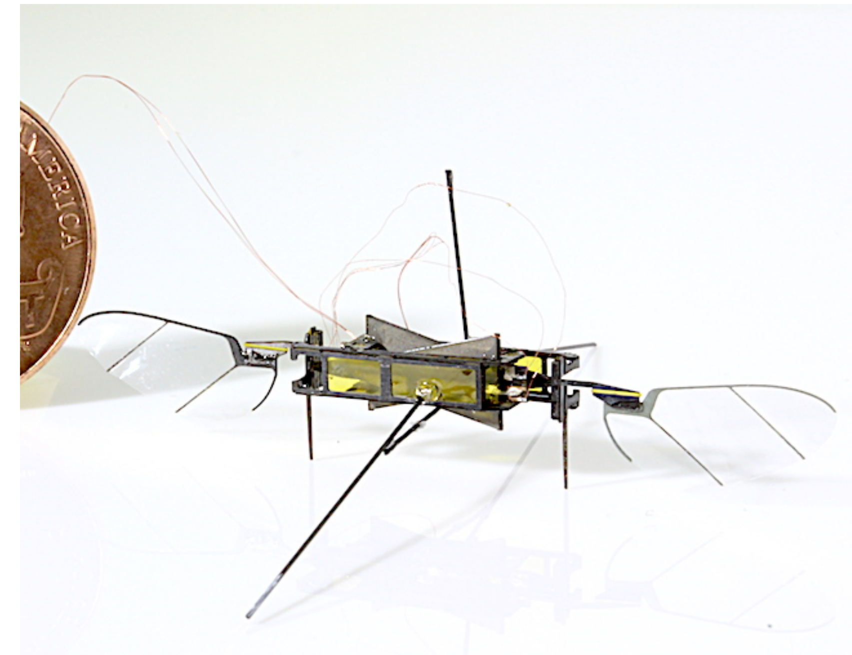
quad-rotor actuation

actuation with two wings





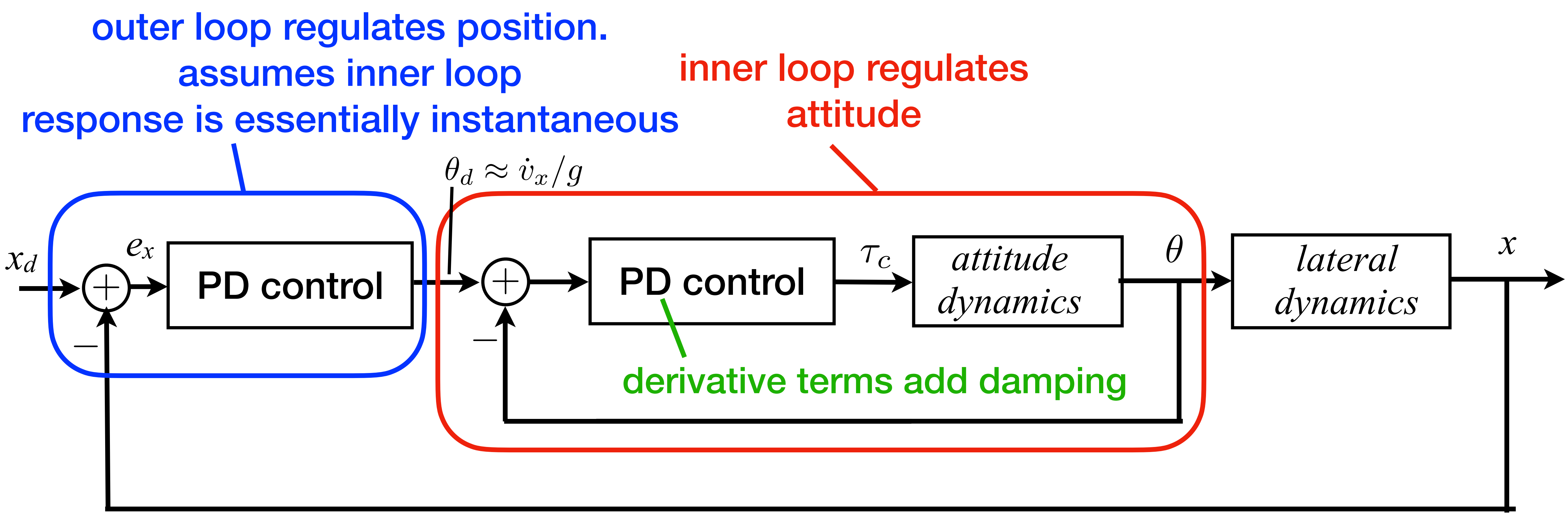




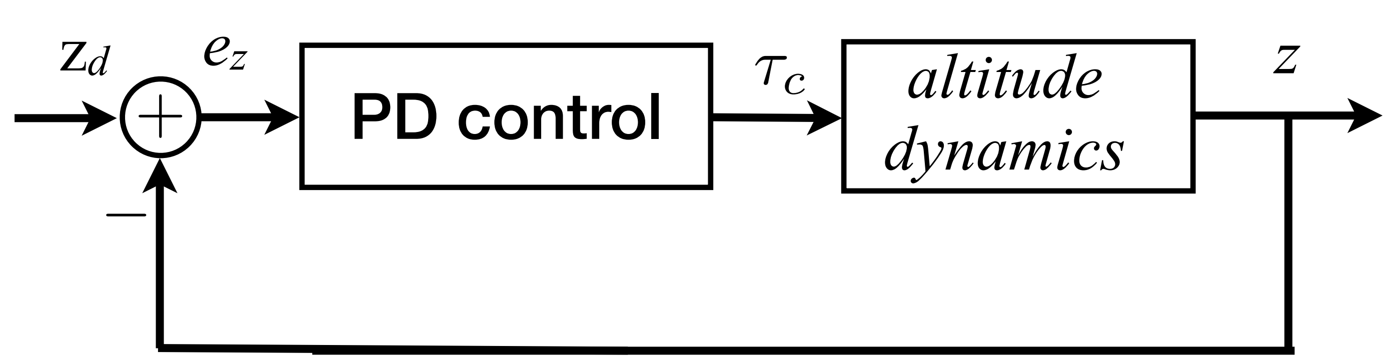
- two rotors spin one direction and two in the other direction

- vary angle and amplitude of flapping wings

insight into flight control: One approach is nested loops (problem set 2)



- plus a separate, independent altitude controller:



more systematic and modern approach

1. A good model: Newton-Euler equations of Motion

$$\Sigma \boldsymbol{f} = m\dot{\boldsymbol{v}}$$

$$\Sigma \boldsymbol{\tau} = \mathbf{J}\dot{\boldsymbol{\omega}} + \boldsymbol{\omega} \times \mathbf{J}\boldsymbol{\omega}$$

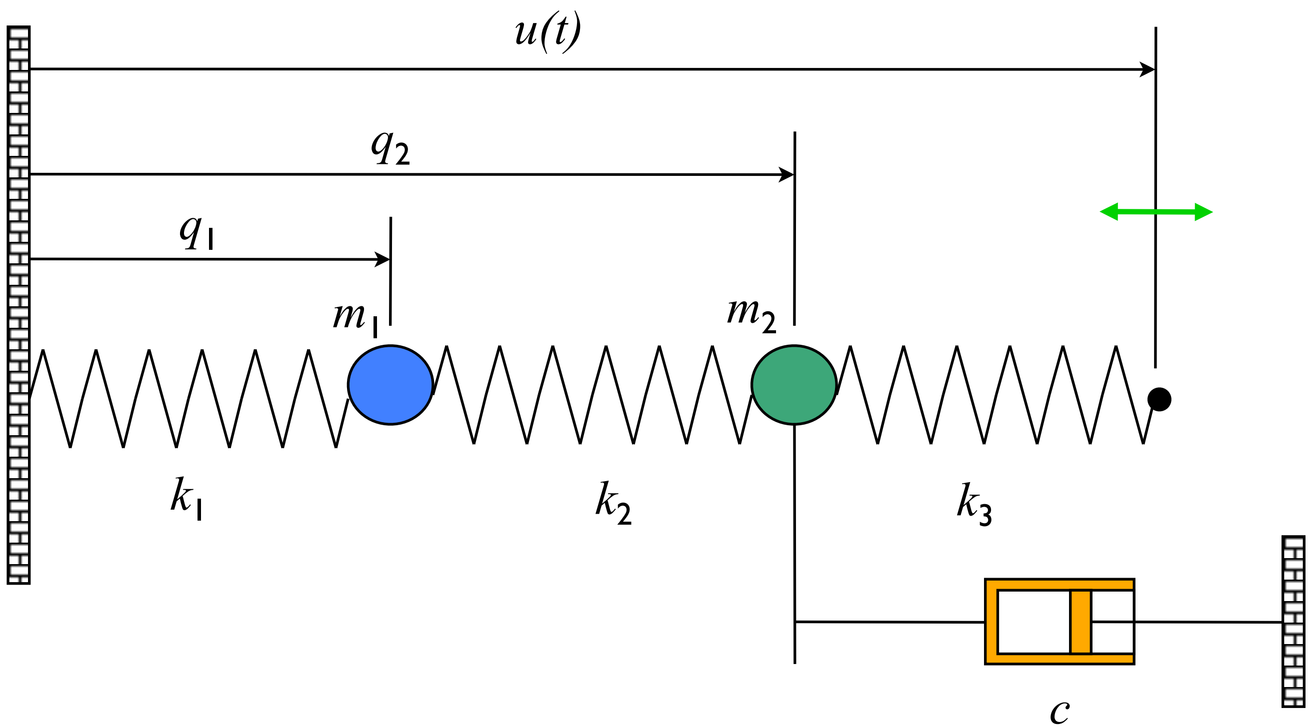
$\boldsymbol{f}, \boldsymbol{\tau}$	**force and torque**
$\boldsymbol{v}, \boldsymbol{\omega}$	**linear, angular velocity**
$\mathbf{J}$	**moment of inertia matrix**

- this is a *nonlinear system.*
- we will control it with *linear feedback controller*
- will return to these equations in more detail next week

2. Optimal control: measure performance with a cost function

Controlling nonlinear systems using linear state-space control

State-space model example: a Spring Mass System



Model: rigid body physics

- Sum of forces = mass * acceleration
- Hooke's law: $F = k(x - x_{\text{rest}})$
- Viscous friction: $F = c\,v$

$$\boxed{\begin{aligned} m_1\ddot{q}_1 &= k_2(q_2 - q_1) - k_1 q_1 \\ m_2\ddot{q}_2 &= k_3(u - q_2) - k_2(q_2 - q_1) - c\dot{q}_2 \end{aligned}}$$

Converting models to state space form

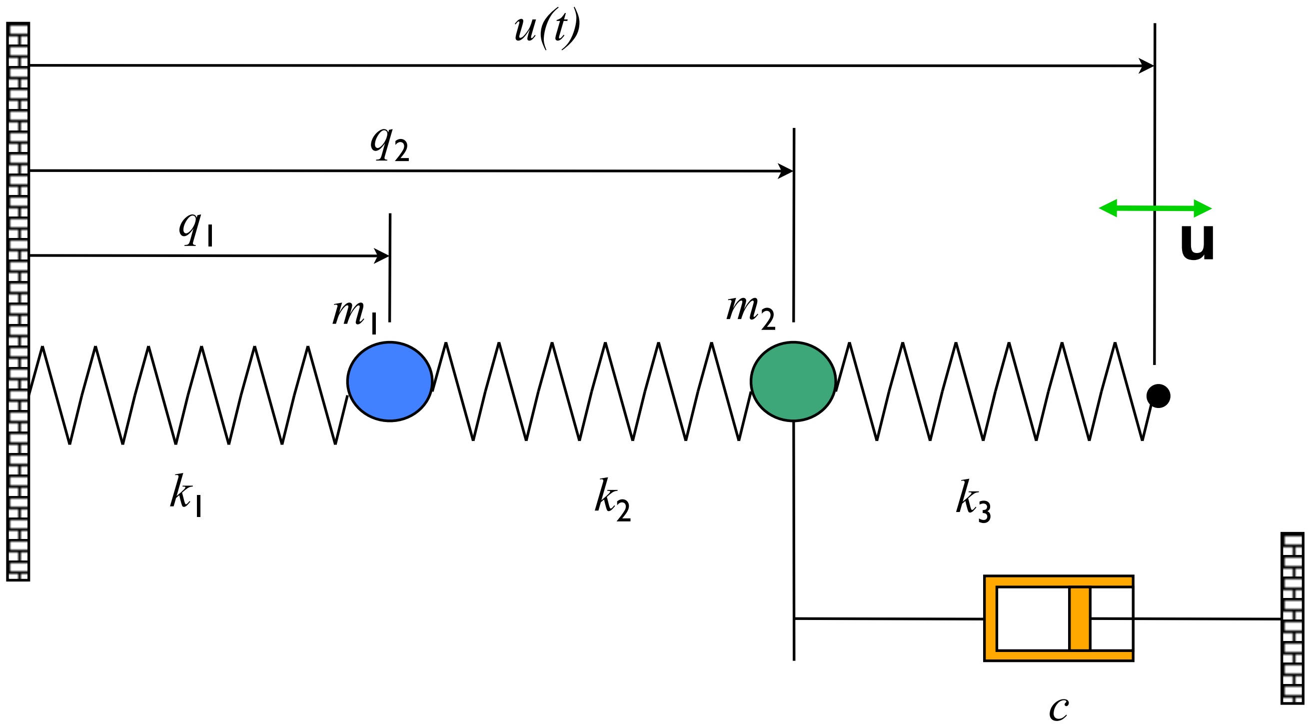
- Construct a *vector* of the variables that are required to specify the evolution of the system
- Write dynamics as a *system* of first order differential equations:

$$\frac{d}{dt}\begin{bmatrix} q_1 \\ q_2 \\ \dot{q}_1 \\ \dot{q}_2 \end{bmatrix} = \begin{bmatrix} \dot{q}_1 \\ \dot{q}_2 \\ \frac{k_2}{m}(q_2 - q_1) - \frac{k_1}{m} q_1 \\ \frac{k_3}{m}(u - q_2) - \frac{k_2}{m}(q_2 - q_1) - \frac{c}{m}\dot{q} \end{bmatrix}$$

$$y = \begin{bmatrix} q_1 \\ q_2 \end{bmatrix}$$

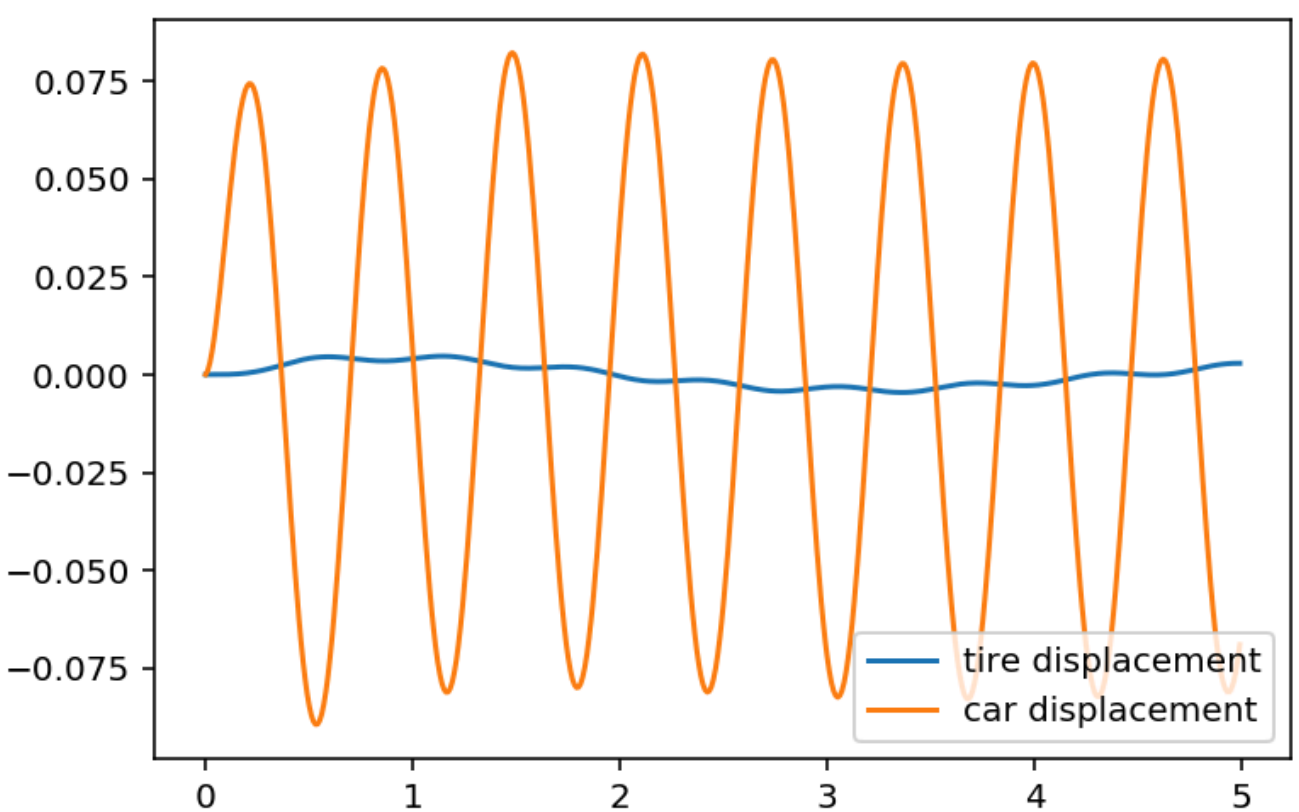
"State space form"

Simulating a state-space system



basic task: repeatedly calculate state update:

$$\boldsymbol{q}_{t+\Delta T} = \boldsymbol{q}_t + \Delta T \dot{\boldsymbol{q}}_t = \boldsymbol{q}_t + \Delta T \boldsymbol{f}(\boldsymbol{q})$$



Python simulation

```
import numpy as np
import matplotlib.pyplot as plt

k1 = k2 = k3 = m1 = c = 1
m2 = 0.1
dt = 0.01

time = np.arange(0, 5, dt)
q_data = np.zeros((len(time), 4))
q = np.array((0, 0, 0, 0))      ← initial condition
def f(q, u):                     ← dynamics function
     return np.array((
          q[2],
          q[3],
          -(k1+k2)/m1*q[0] + k2/m1*q[1],
          k2/m2*q[0] - (k2+k3)/
              m2*q[1] - c/m2*q[3] + k3/m2*u))

for idx, t in enumerate(time):
    u = np.cos(10*t)
    q = q + dt * f(q, u)         ← update step
    q_data[idx,:] = q            ← store result
plt.plot(time, q_data[:,0:2])
plt.legend(('car displacement (q1)',
            'tire displacement (q2)'))
```

general form of differential equations

State space form

$$\frac{dx}{dt} = f(x,u) \qquad y = h(x,u)$$

General form

$$\frac{dx}{dt} = Ax + Bu \qquad y = Cx + Du$$

Linear system

$$x \in \mathbb{R}^n,\ u \in \mathbb{R}^p \qquad y \in \mathbb{R}^q$$

• x = state; nth order

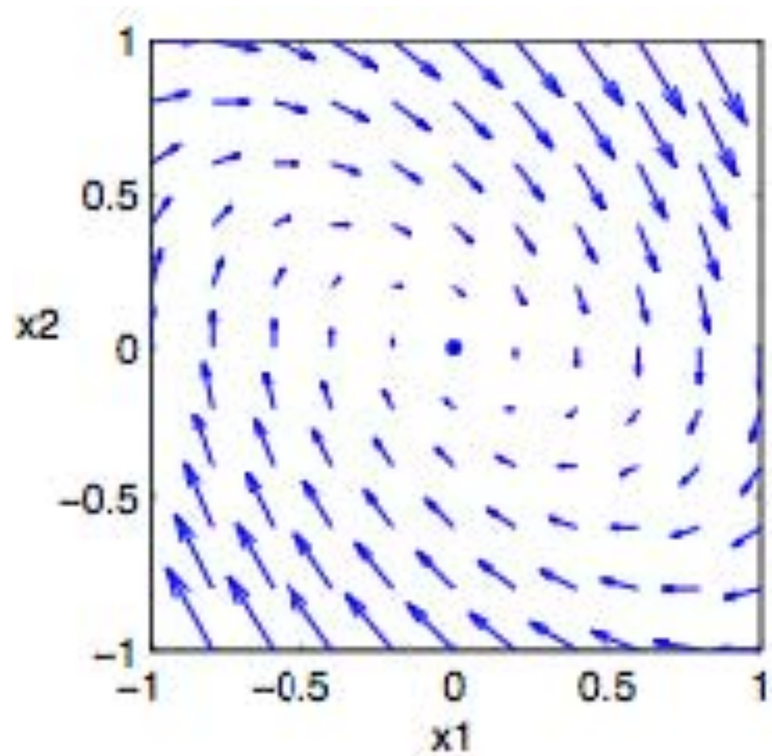
phase plots show 2D behavior

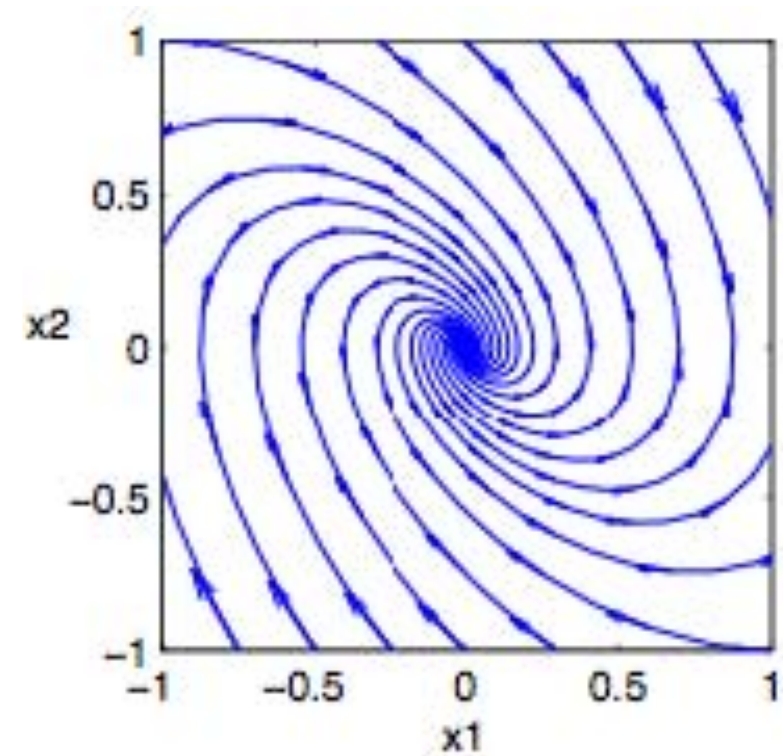
Phase plane plots show 2D dynamics as *vector fields* & *stream functions*

- $\dot{x} = f(x, u(x)) = F(x)$
- Plot $F(x)$ as a vector on the plane; stream lines follow the flow of the arrows

$$\frac{d}{dt}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_2 \\ -x_1 - x_2 \end{bmatrix}$$

python: use 'streamplot' function in Matplotlib





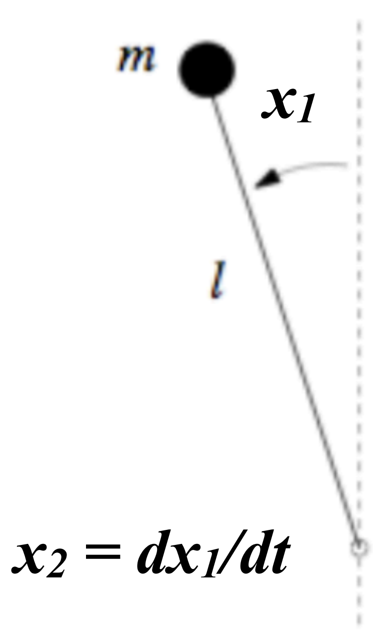
equilibrium points

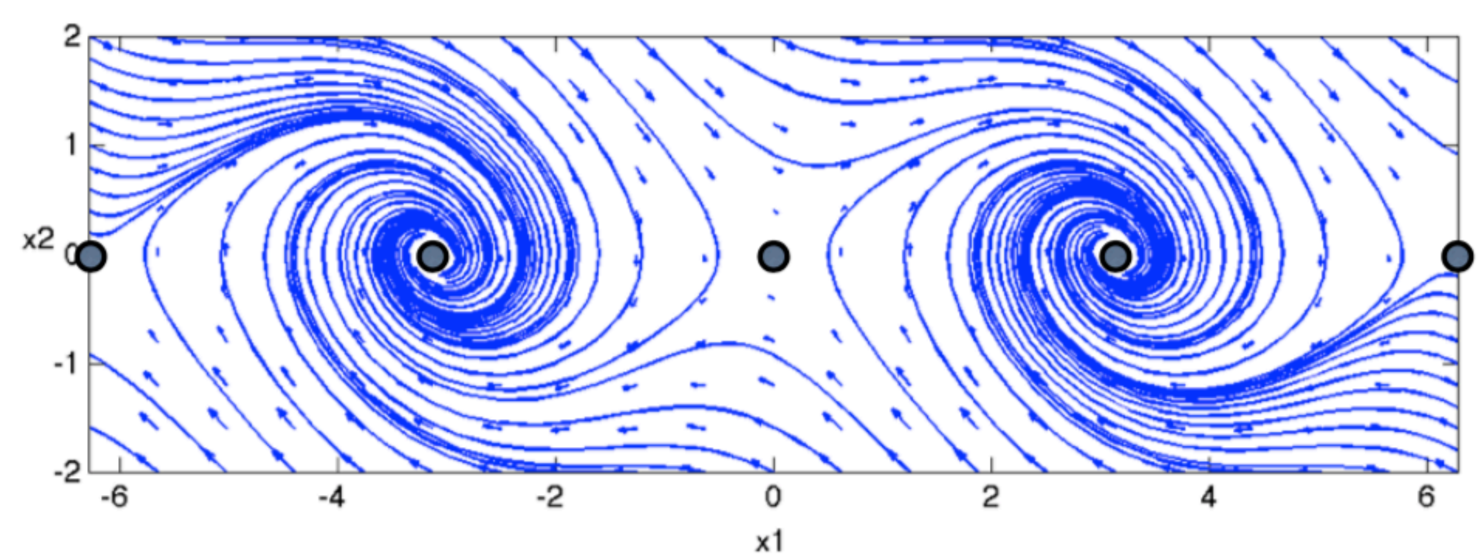
Equilibrium points represent stationary conditions for the dynamics

The *equilibria* of the system $\dot{x} = f(x)$ are the points x_e such that $f(x_e) = 0$.

Example:

$$\frac{dx}{dt} = \begin{bmatrix} x_2 \\ \sin x_1 - \gamma x_2 \end{bmatrix} \qquad \Rightarrow \qquad x_e = \begin{bmatrix} \pm n\pi \\ 0 \end{bmatrix}$$



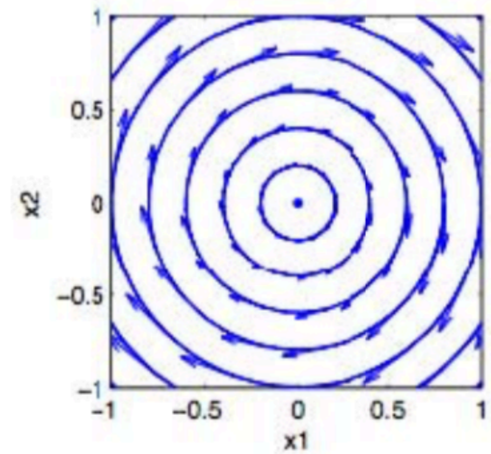


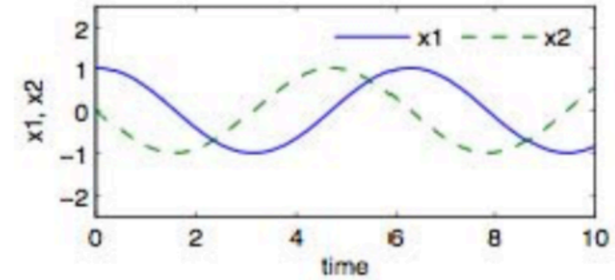
stability of equilibrium points

An equilibrium point is:

Stable if initial conditions that start near the equilibrium point, stay near

- Also called “stable in the sense of Lyapunov

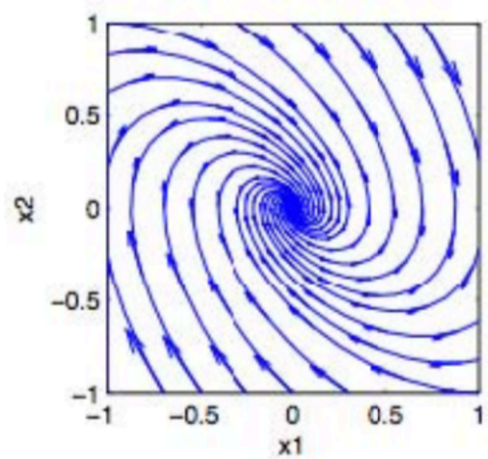


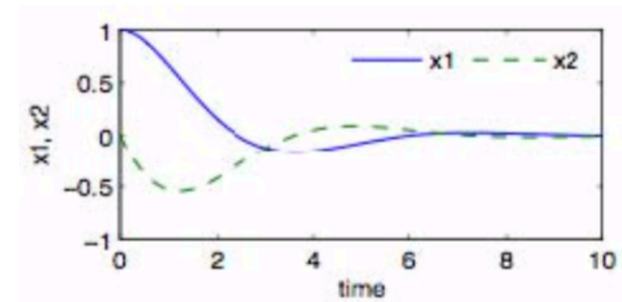


“stable” but not asymptotically stable

Asymptotically stable if all nearby initial conditions converge to the equilibrium point

- Stable + converging

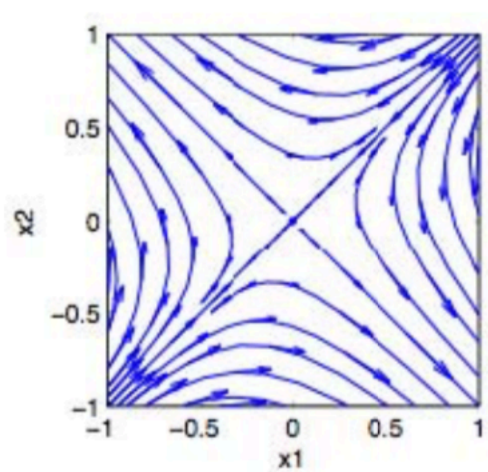


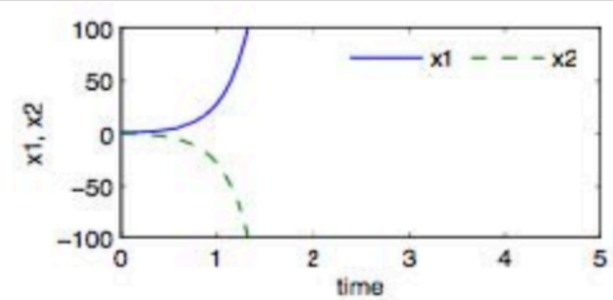


asymptotically stable

Unstable if some initial conditions diverge from the equilibrium point

- May still be some initial conditions that converge



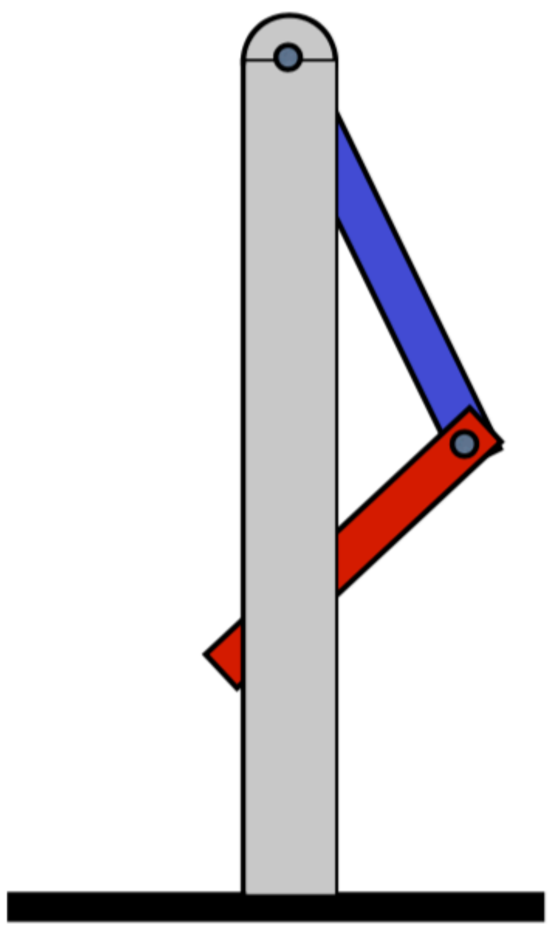


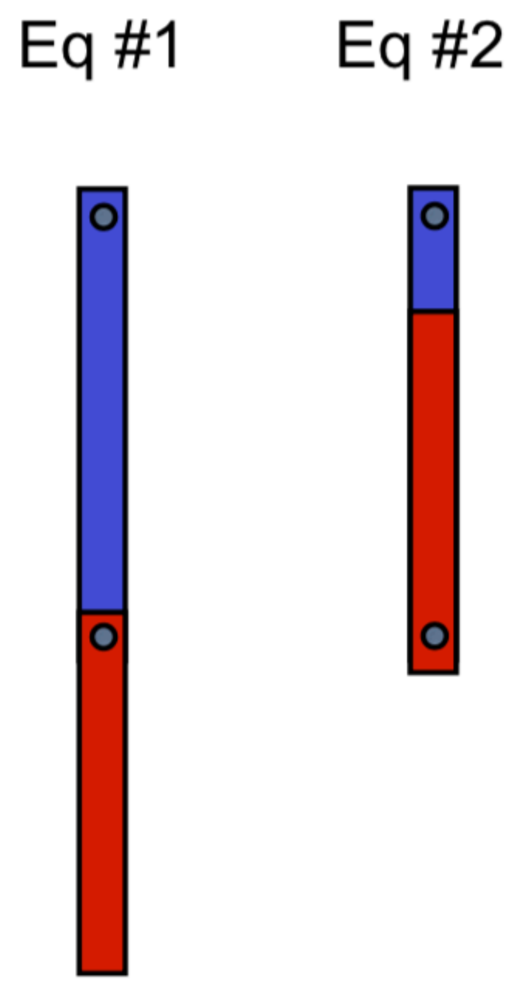
unstable

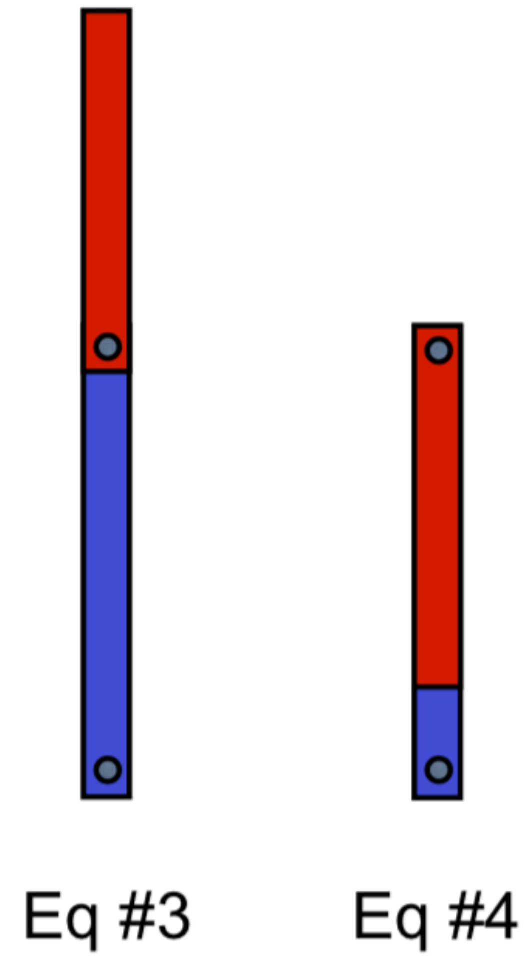
Example #1: Double Inverted Pendulum

Two series coupled pendula

- States: pendulum angles (2), velocities (2)
- Dynamics: *F* = *ma* (balance of forces)
- Dynamics are very nonlinear







Stability of equilibria

- Eq #1 is stable
- Eq #3 is unstable
- Eq #2 and #4 are unstable, but with some stable "modes"

Linearization about an equilibrium point

$$\dot{x} = f(x,u) \qquad \longrightarrow \qquad \dot{z} = Az + Bv$$
$$y = h(x,u) \qquad \qquad \qquad w = Cz + Dv$$

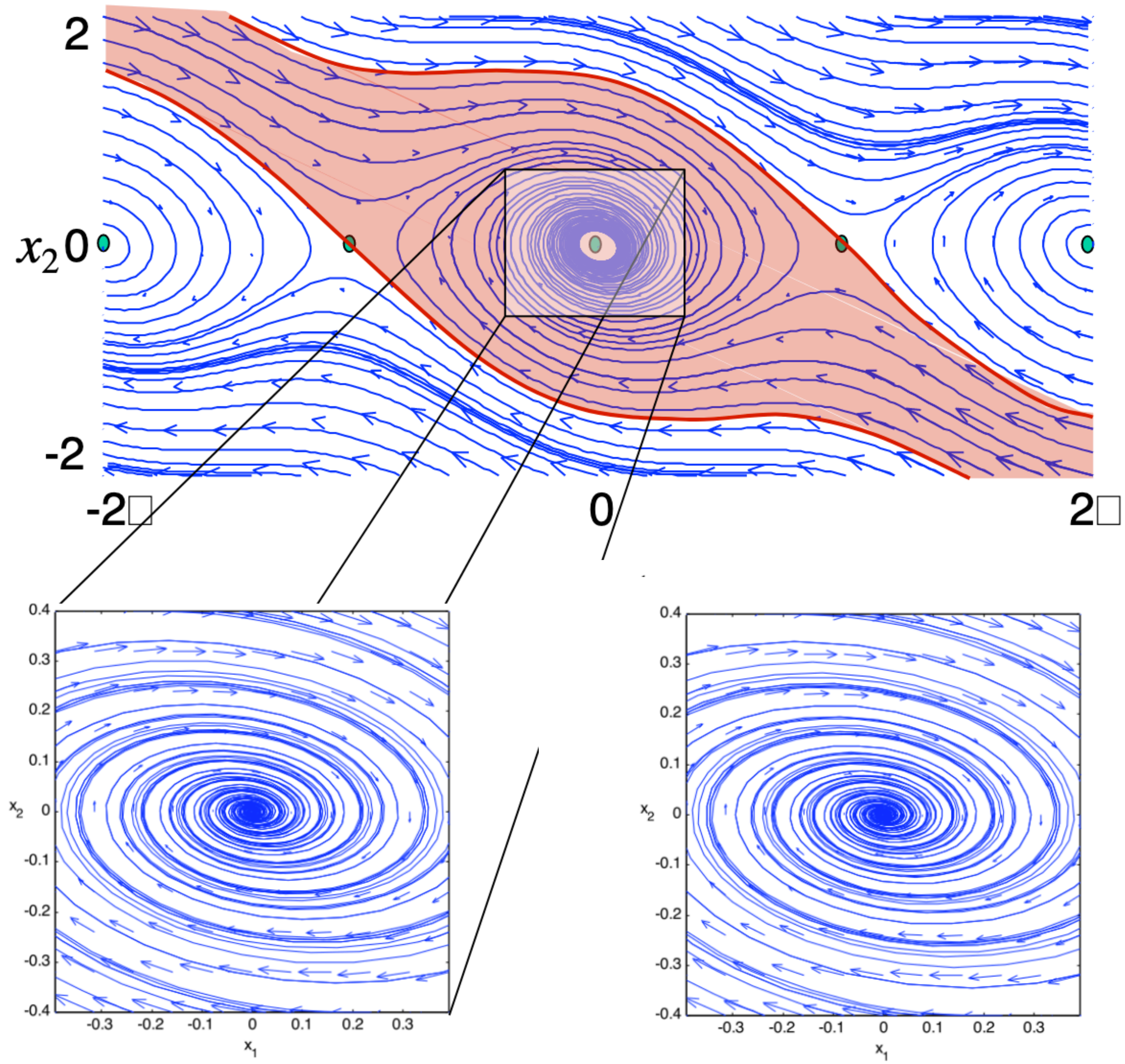
to "linearize" around $x = x_e$:

1. find x_e, u_e such that $f = 0$
2. define $y_e = h(x_e, u_e)$

   $z = x - x_e \quad v = u - u_e \quad w = y - y_e$
3. then

$$A = \left.\frac{\partial f}{\partial x}\right|_{(x_e,u_e)} \qquad B = \left.\frac{\partial f}{\partial u}\right|_{(x_e,u_e)}$$

$$C = \left.\frac{\partial h}{\partial x}\right|_{(x_e,u_e)} \qquad D = \left.\frac{\partial h}{\partial u}\right|_{(x_e,u_e)}$$



full nonlinear model **linear model (honest!)**

Remarks

- In examples, this is often equivalent to small angle approximations, etc
- Only works *near* to equilibrium point
  - use linearization to design controller

big idea: if combined linearized system + controller is stable $\Longrightarrow$ nonlinear system (incl control) is stable nearby

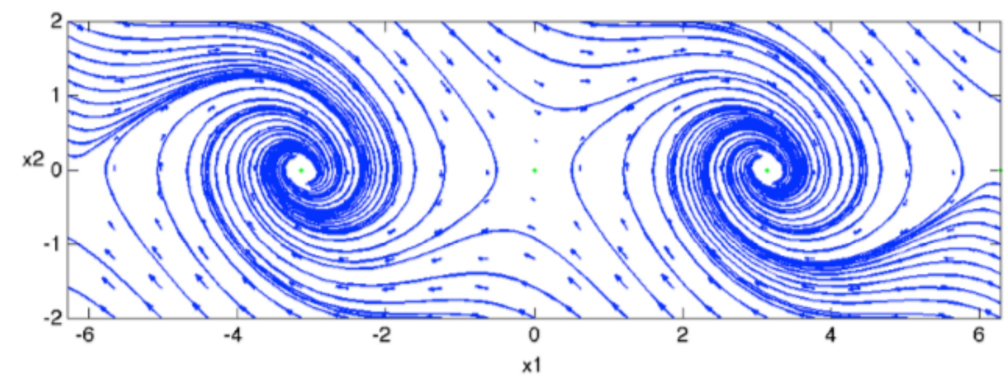
Jacobian linearization matrix

$$A = \left.\frac{\partial f}{\partial x}\right|_{(x_e, u_e)} = \left.\begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \cdots & \frac{\partial f_1}{\partial x_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial f_m}{\partial x_1} & \cdots & \frac{\partial f_m}{\partial x_n} \end{bmatrix}\right|_{(\boldsymbol{x_e}, \boldsymbol{u_e})}$$

Example: Stability Analysis of Inverted Pendulum

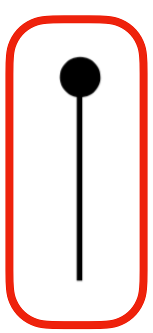
System dynamics

$$\frac{dx}{dt} = \begin{bmatrix} x_2 \\ \sin x_1 - \gamma x_2 \end{bmatrix},$$



Equilibria: where $\dot{x} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow x_e = \begin{bmatrix} \pm\pi k, \quad k = 0,\ 1,\ 2,\ 3\ldots \\ 0 \end{bmatrix}$

Linearize to assess stability: $\frac{\partial f}{\partial x} = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ \cos x_1 & -\gamma \end{bmatrix}$

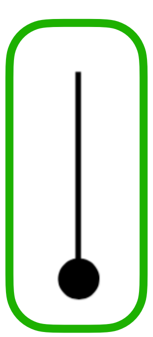


Upward equilibria: $x_1 = \pm 2\pi k, \quad k = 0,\ 1,\ 2,\ 3\ldots$

$$A = \frac{\partial f}{\partial x}\Big|_{x_e} = \begin{bmatrix} 0 & 1 \\ 1 & -\gamma \end{bmatrix}$$

eigenvalues: $\lambda = -\frac{1}{2}\gamma \pm \frac{1}{2}\sqrt{4+\gamma^2}$

for $\gamma = \mathbf{0.1},\ \lambda \cong (\mathbf{0.95}, -\mathbf{1.05}) \Longrightarrow$ **unstable**



Downward equilibria: $x_1 = \pi \pm 2\pi k, \quad k = 0,\ 1,\ 2,\ 3\ldots$

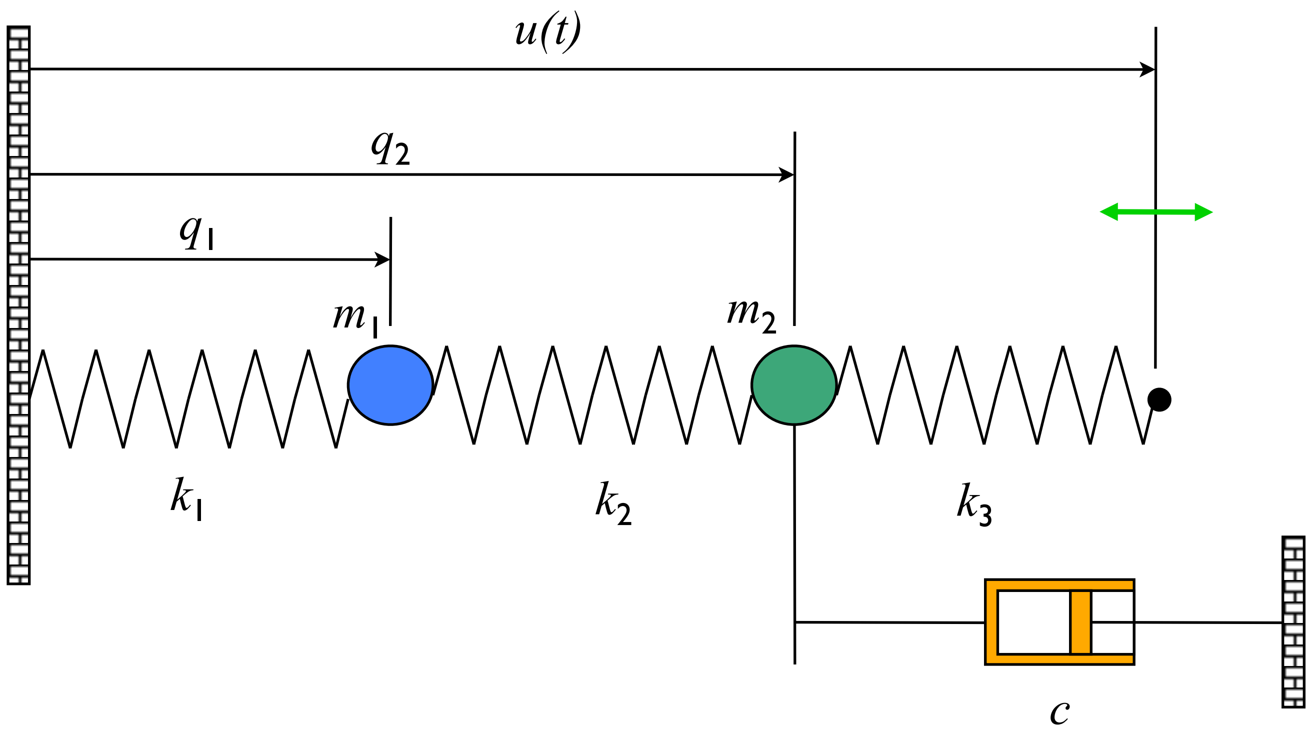
use $z_1 = x_1 - x_{1e} = x_1 - \pi, \quad z_2 = x_2 \quad \Rightarrow \dot{z} = Az$

$$A = \frac{\partial f}{\partial x}\Big|_{x_e} = \begin{bmatrix} 0 & 1 \\ -1 & -\gamma \end{bmatrix}$$

eigenvalues: $\lambda = -\frac{1}{2}\gamma \pm \frac{1}{2}\sqrt{-4+\gamma^2}$

for $\gamma = \mathbf{0.1},\ \lambda \cong (-\mathbf{0.05}+i, -\mathbf{0.05}-i) \Longrightarrow$ **stable**

example 2: matrix representation of a linear system



Model: rigid body physics

- Sum of forces = mass * acceleration
- Hooke's law: $F = k(x - x_{rest})$
- Viscous friction: $F = c\, v$

$$\boxed{\begin{aligned} m_1\ddot{q}_1 &= k_2(q_2 - q_1) - k_1 q_1 \\ m_2\ddot{q}_2 &= k_3(u - q_2) - k_2(q_2 - q_1) - c\dot{q}_2 \end{aligned}}$$

$$\frac{d}{dt}\begin{bmatrix} q_1 \\ q_2 \\ \dot{q}_1 \\ \dot{q}_2 \end{bmatrix} = \begin{bmatrix} \dot{q}_1 \\ \dot{q}_2 \\ \frac{k_2}{m}(q_2 - q_1) - \frac{k_1}{m}q_1 \\ \frac{k_3}{m}(u - q_2) - \frac{k_2}{m}(q_2 - q_1) - \frac{c}{m}\dot{q} \end{bmatrix}$$

$$y = \begin{bmatrix} q_1 \\ q_2 \end{bmatrix}$$

"State space form"

Matrix representation:

$$\dot{x} = Ax + Bu$$

$$\dot{x} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -\frac{k_1 + k_2}{m} & \frac{k_2}{m} & 0 & 0 \\ -\frac{k_2}{m} & -\frac{k_2 + k_3}{m} & 0 & 0 \end{bmatrix} x + \begin{bmatrix} 0 \\ 0 \\ 0 \\ \frac{k_3}{m} \end{bmatrix} u$$

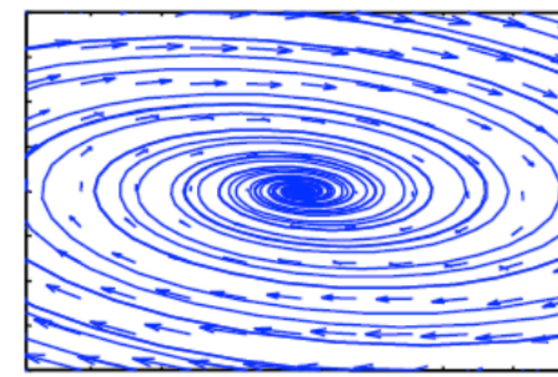
$$y = [1 \quad 1 \quad 0 \quad 0]x = Cx$$

State Space Control Design Concepts

System description: single input, single output system (MIMO also OK)

$$\begin{aligned} \dot{x} &= f(x,u) \qquad & x \in \mathbb{R}^n,\ x(0) \text{ given} \\ y &= h(x) \qquad & u \in \mathbb{R},\ y \in \mathbb{R} \end{aligned}$$

Stability: stabilize the system around an equilibrium point



- Given equilibrium point $x_e \in \mathbb{R}^n$, find control "law" $u = \alpha(x)$ such that

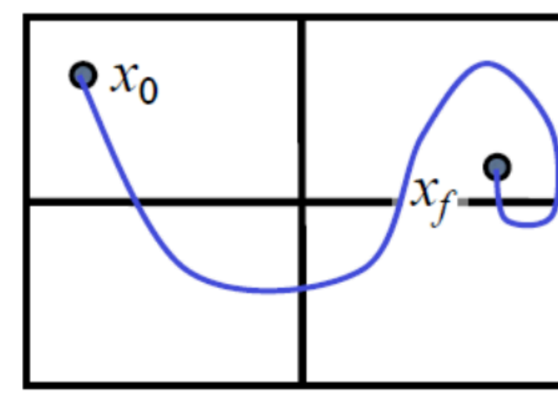
$$\lim_{t\to\infty} x(t) = x_e \text{ for all } x(0) \in \mathbb{R}^n$$

- Often choose x_e so that $y_e = h(x_e)$ has desired value r (constant)

Reachability: steer the system between two points

- Given $x_o, x_f \in \mathbb{R}^n$, find an input $u(t)$ such that

$$\dot{x} = f(x,u(t)) \text{ takes } x(t_0) = x_0 \to x(T) = x_f$$



Tests for Reachability

$$\dot{x} = Ax + Bu \qquad x \in \mathbb{R}^n,\ x(0) \text{ given}$$
$$y = Cx \qquad u \in \mathbb{R},\ y \in \mathbb{R}$$

$$x(T) = e^{AT}x_0 + \int_{\tau=0}^{T} e^{A(T-\tau)}Bu(\tau)d\tau$$

Thm A linear system is reachable if and only if the $n \times n$ *reachability matrix*

$$\begin{bmatrix} B & AB & A^2B & \cdots & A^{n-1}B \end{bmatrix}$$

is full rank.

Note: also called "controllability" matrix

Remarks

- *Very* simple test : `control.ctrb(A,B)` and check rank with `numpy.linalg.matrix_rank()`
- If this test is satisfied, we say "the pair (A,B) is reachable"

State space controller design for linear systems

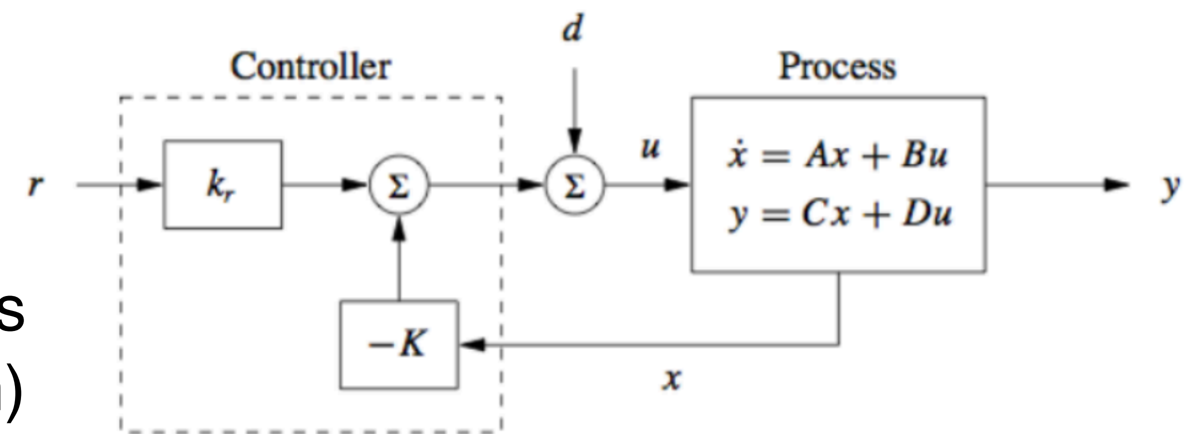
$$\dot{x} = Ax + Bu \qquad x \in \mathbb{R}^n,\ x(0) \text{ given}$$
$$y = Cx \qquad u \in \mathbb{R},\ y \in \mathbb{R}$$

$$x(T) = e^{AT}x_0 + \int_{\tau=0}^{T} e^{A(T-\tau)}Bu(\tau)\,d\tau$$

Goal: find a linear control law $u = -Kx$ such that the closed loop system

$$\dot{x} = Ax + Bu = (A - BK)x$$

is stable at $x = 0$ (assumes x are coordinates relative to location of equilibrium)



- Stability based on eigenvalues ⇒ use K to make eigenvalues of $(A - BK)$ stable
- Can also link eigenvalues to *performance* (eg, initial condition response)
- Question: when can we place the eigenvalues anyplace that we want?

Theorem The eigenvalues of $(A - BK)$ can be set to arbitrary values if and only if the pair (A, B) is reachable.

Next: one way to choose *K*

control and dynamics

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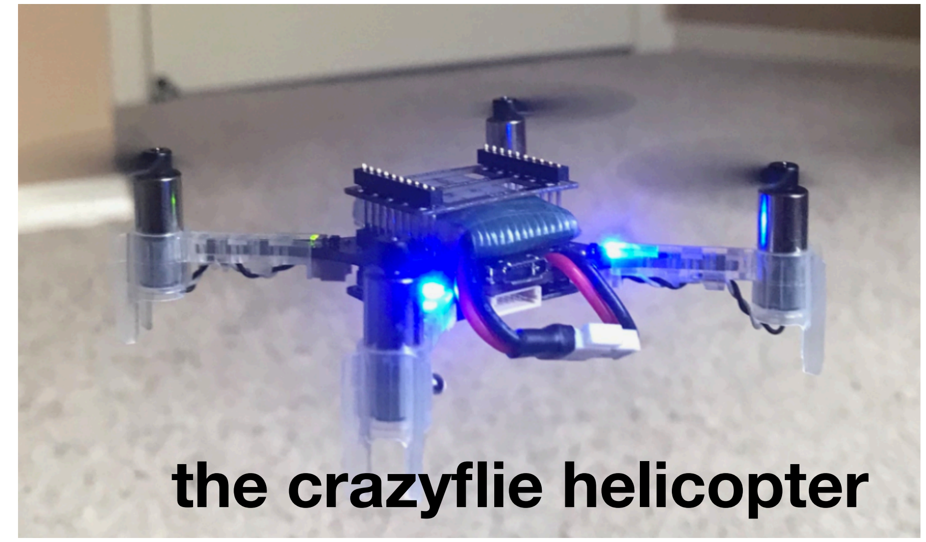
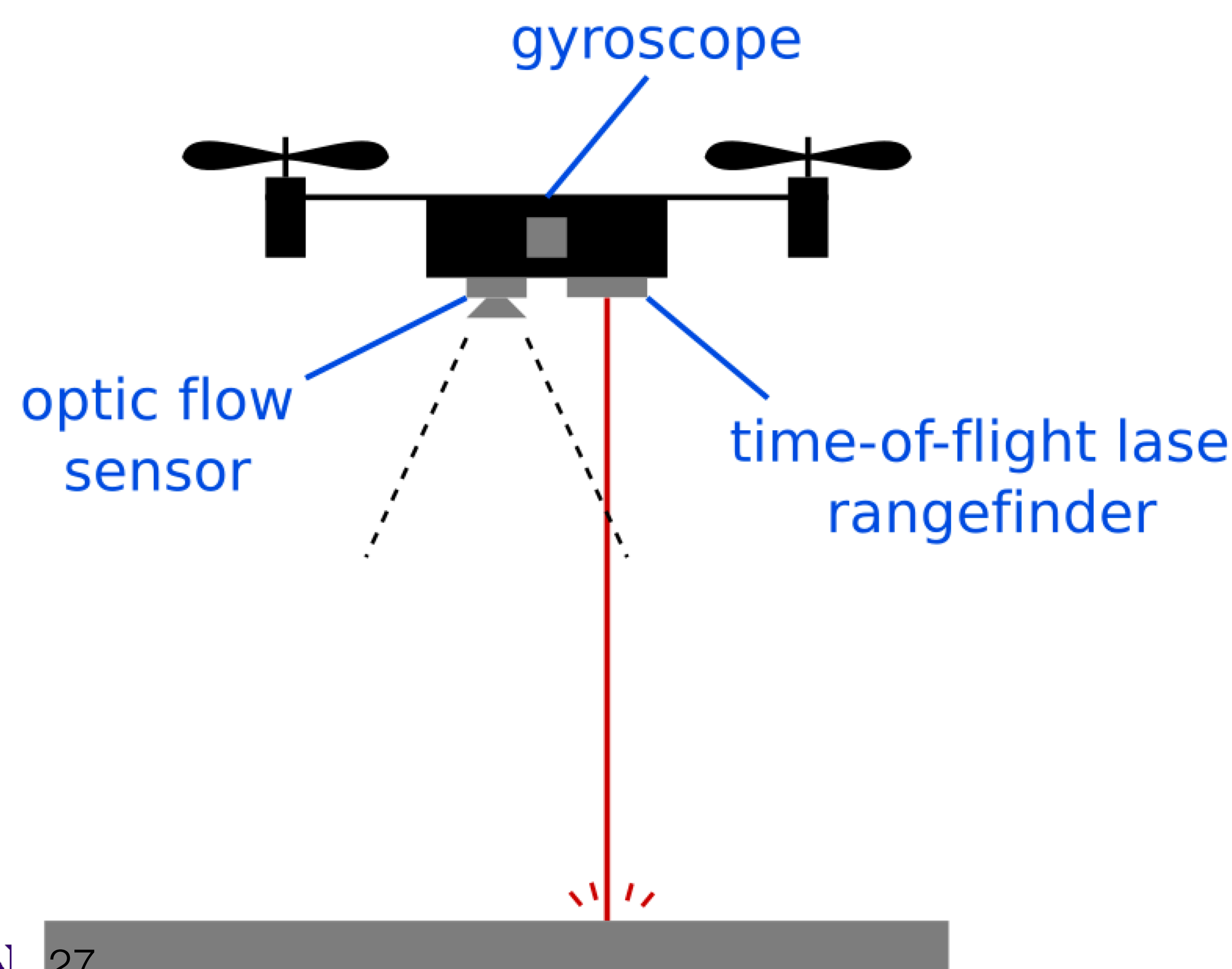
ME 586: Biology-inspired robotics

Project-based portion of this course



- you will work with the crazyflie helicopter as part of your homework problem sets
    - learning objectives:
        - learn basics of robotics and drone control
        - experience implementing bio-inspired control algorithm
- Crazyflie specs:
    - ~30 g, ~4 minute flight time
    - communicates in real-time over bluetooth to laptop
    - sensor suite gives information needed to stabilize and control flight
    - open-source control software



three ideas inspired by biology for how to improve robotics

(the themes of this course)

1. adaptation through evolution and learning

😀 fundamental engineering processes used by biology
☹️ "curse of dimensionality"

2. mechanical intelligence
   - the use of mechanics to reduce or eliminate the need for feedback control
3. parsimony
   - simple and efficient solutions

😀 "shortcut": look directly to biology for inspiration, combine with engineering knowledge

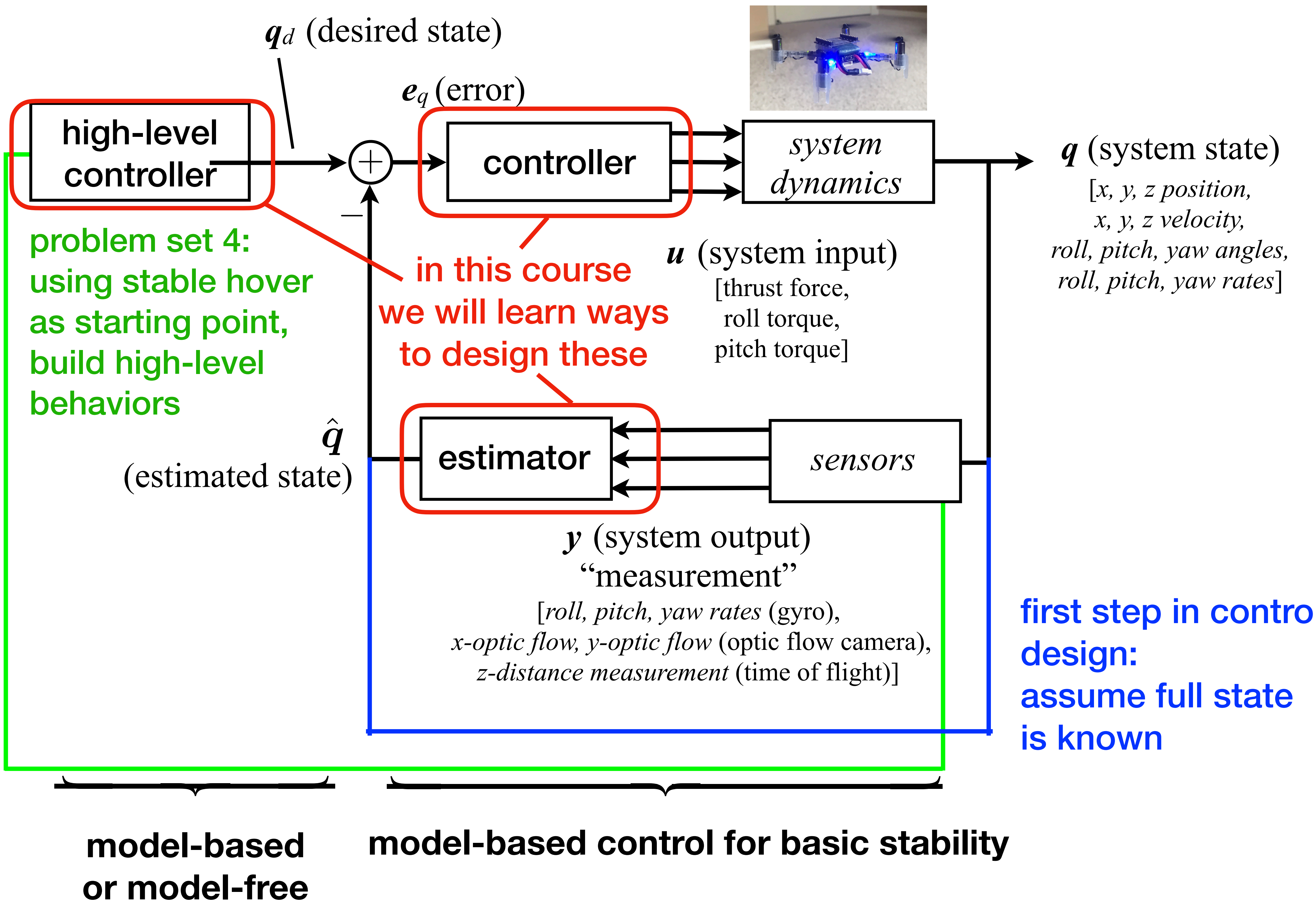
ME586 homework and projects emphasize these. We will show that the optimal control formulation we use for flight stability is also the basis for robot learning.

crazyflie in operation performing odor source localization

Odor Localization

Anderson, Sullivan, Horiuchi, Fuller, & Daniel, *Bioinspiration & Biomimetics* 2020

the controller we will learn



basics: actuation for hovering

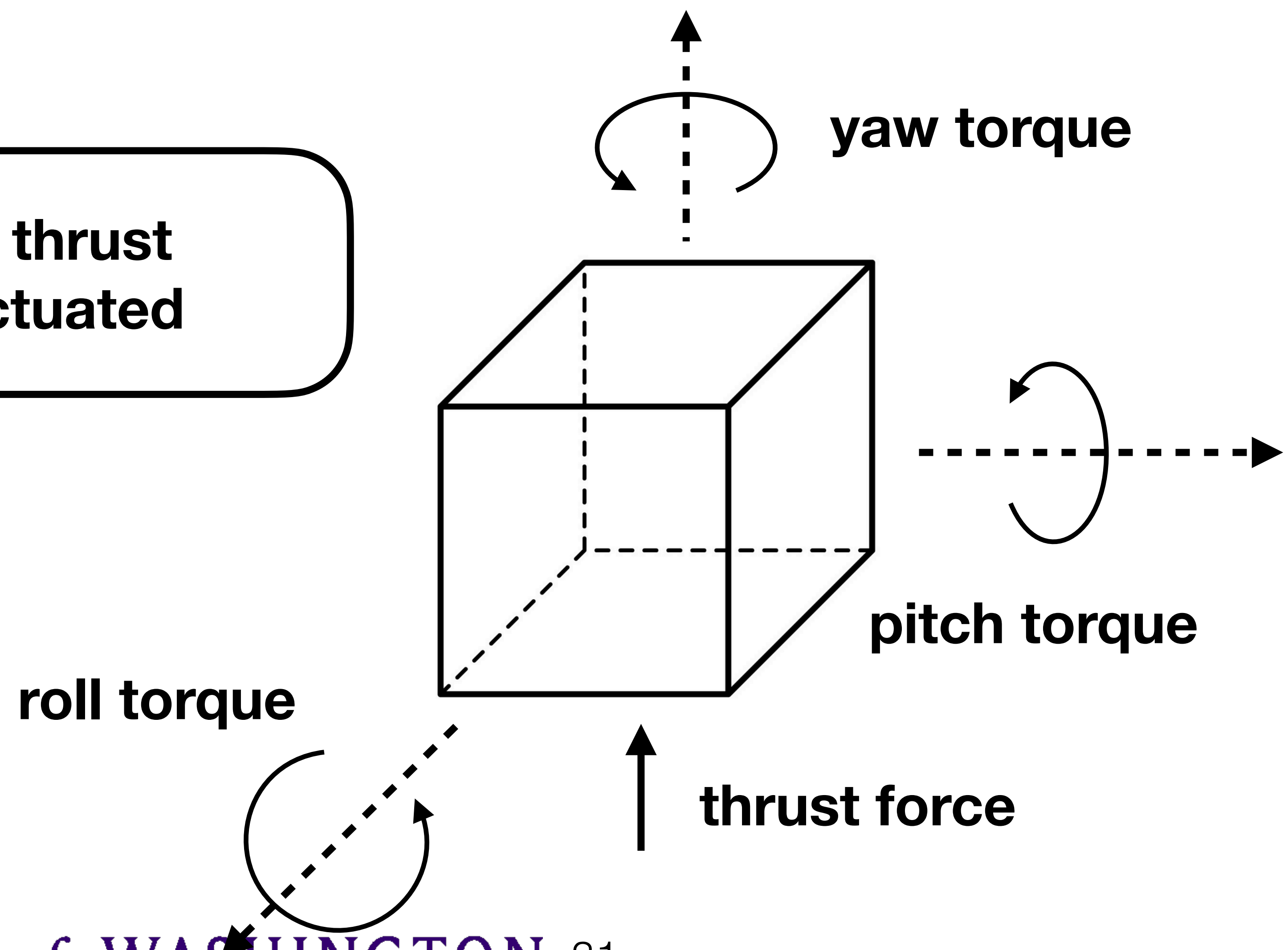
honeybee

single-rotor helicopter

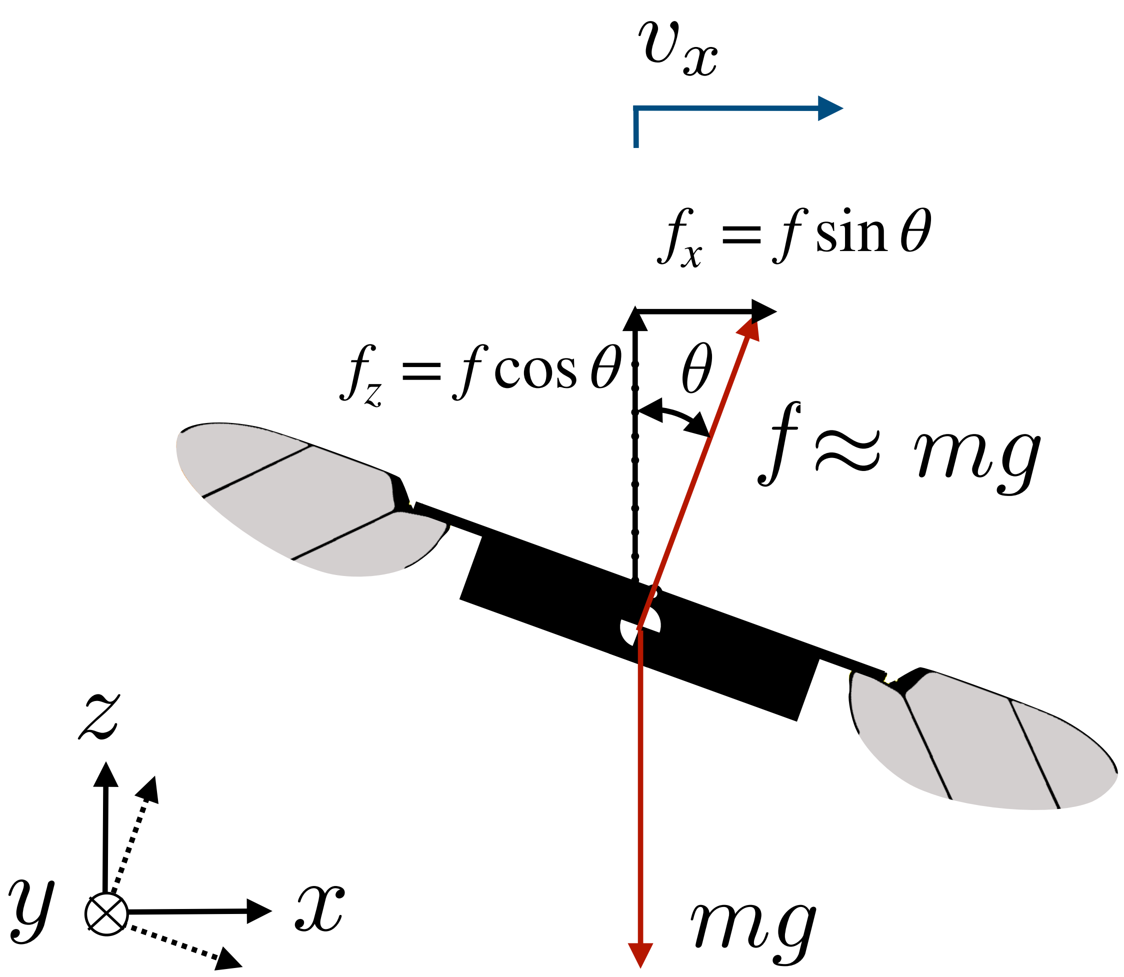
robot flies e.g.
UW Robofly

four-rotor aircraft
“quad-rotors”

typically, lateral thrust
is not directly actuated



If you can tilt, how do you move laterally?



lateral acceleration

$$m\dot{v}_x = f_x = f\sin\theta \approx mg\sin\theta$$

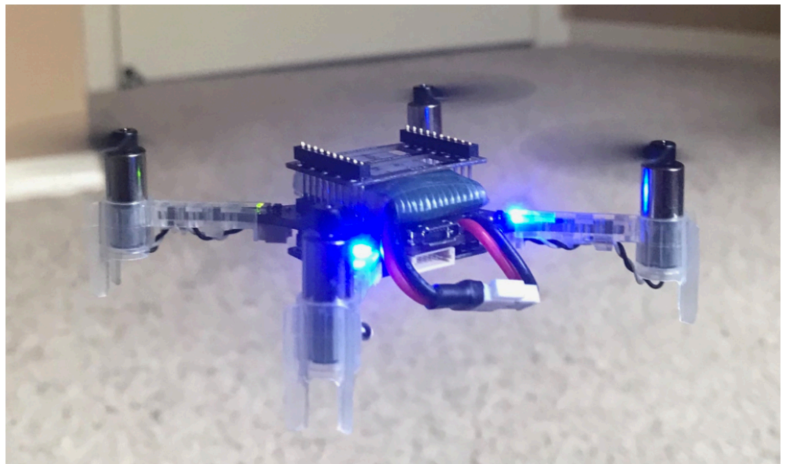
$$\Rightarrow \dot{v}_x = g\sin\theta$$

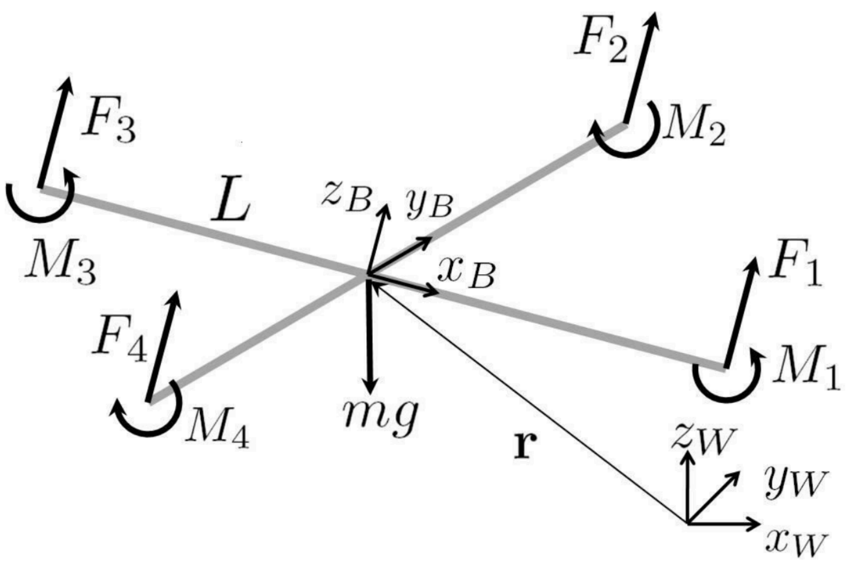
$$\approx g\theta$$ **for small θ**

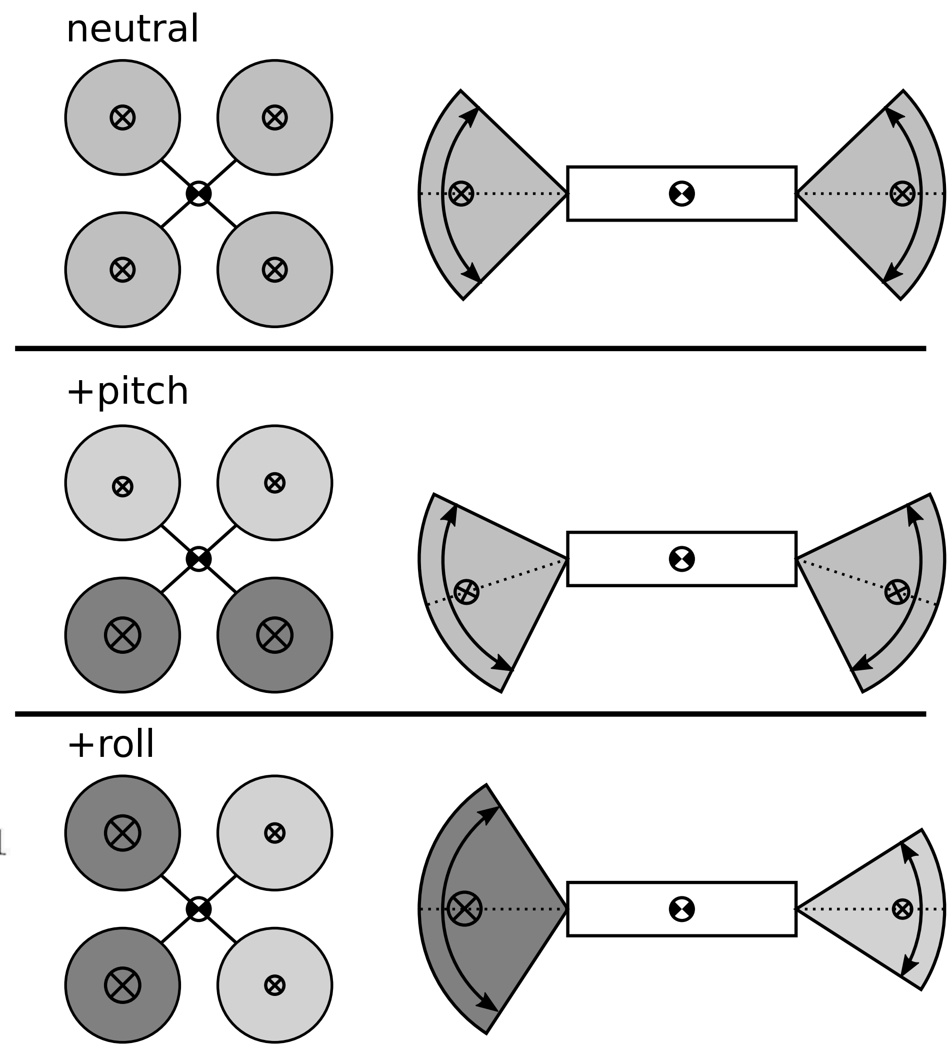
- “helicopter-like” lateral control

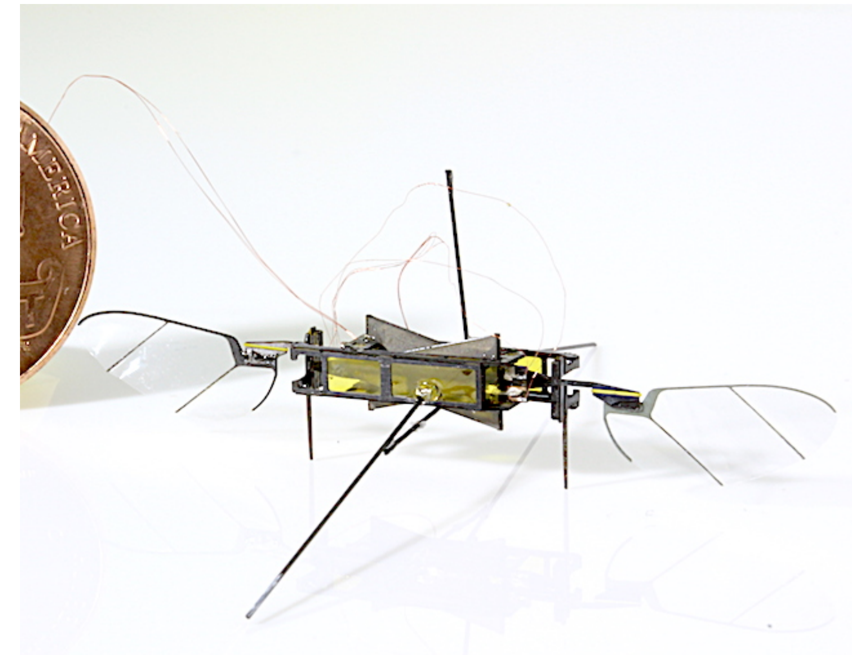
quad-rotor actuation

actuation with two wings





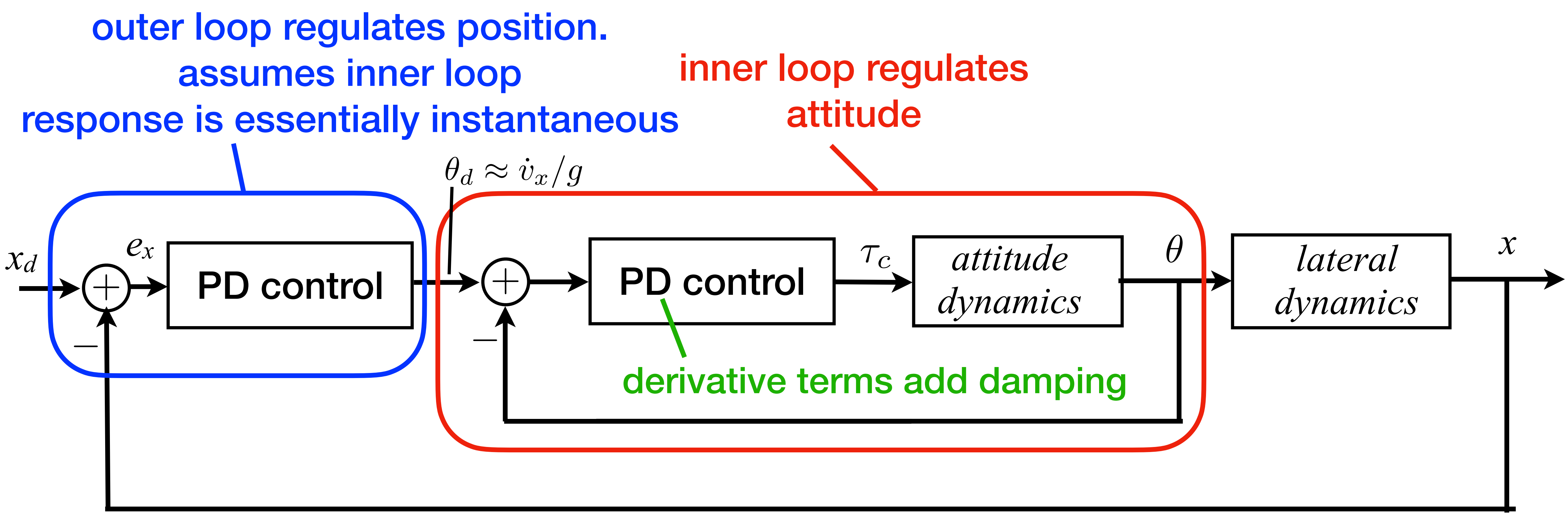




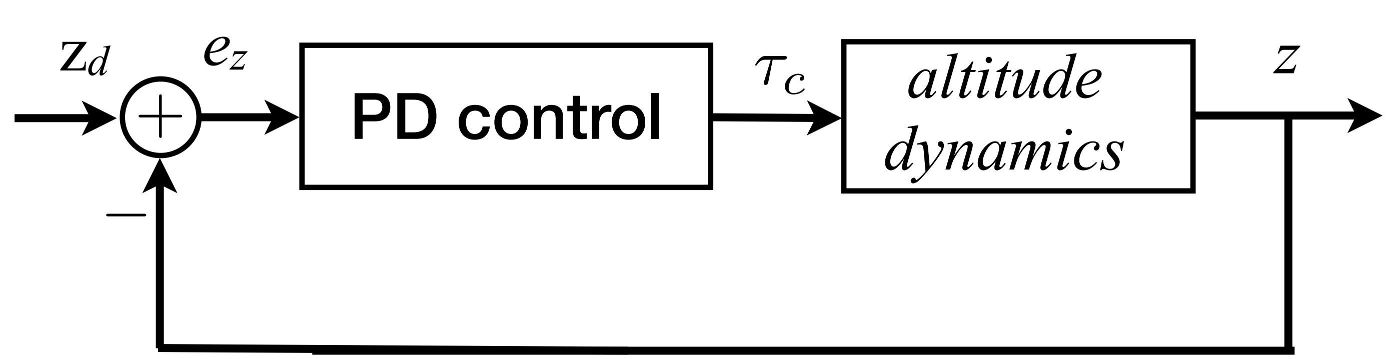
- two rotors spin one direction and two in the other direction

- vary angle and amplitude of flapping wings

insight into flight control: One approach is nested loops (problem set 2)



- plus a separate, independent altitude controller:



more systematic and modern approach

1. A good model: Newton-Euler equations of Motion

$$\Sigma \boldsymbol{f} = m\dot{\boldsymbol{v}}$$

$$\Sigma \boldsymbol{\tau} = \mathbf{J}\dot{\boldsymbol{\omega}} + \boldsymbol{\omega} \times \mathbf{J}\boldsymbol{\omega}$$

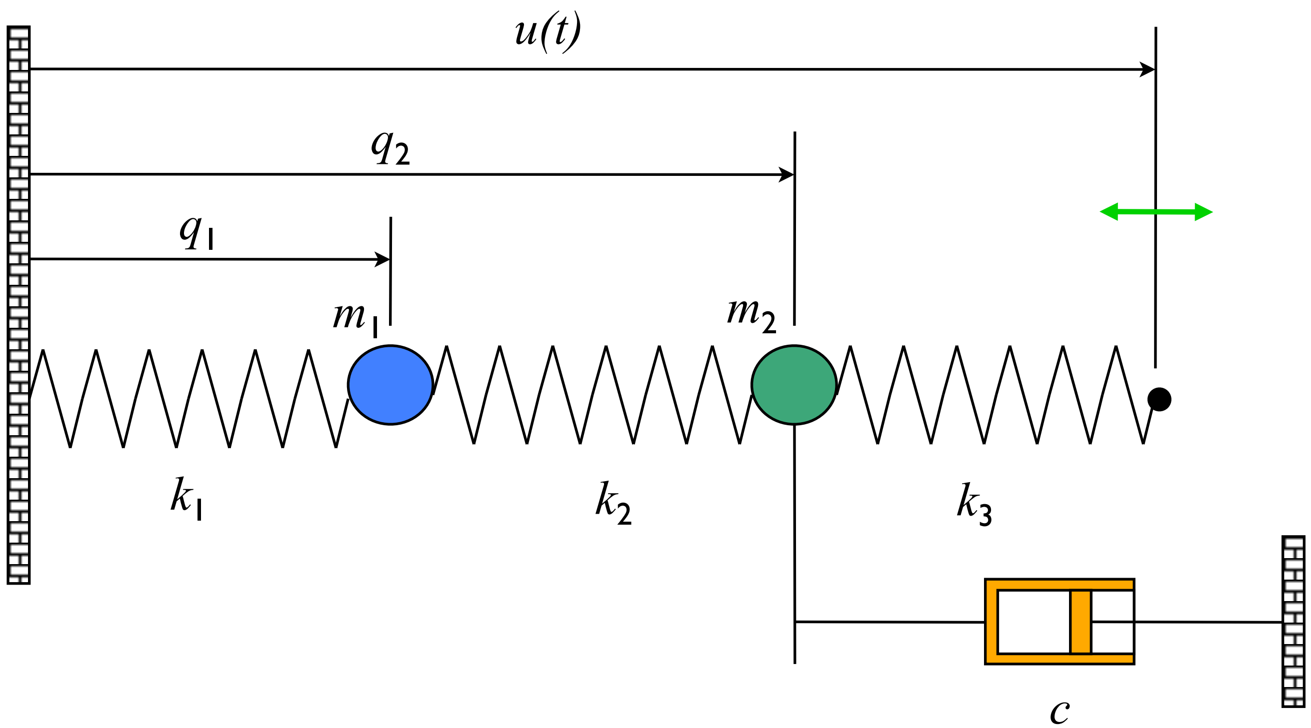
$\boldsymbol{f}, \boldsymbol{\tau}$	**force and torque**
$\boldsymbol{v}, \boldsymbol{\omega}$	**linear, angular velocity**
$\mathbf{J}$	**moment of inertia matrix**

- this is a *nonlinear system.*
- we will control it with *linear feedback controller*
- will return to these equations in more detail next week

2. Optimal control: measure performance with a cost function

Controlling nonlinear systems using linear state-space control

State-space model example: a Spring Mass System



Model: rigid body physics

- Sum of forces = mass * acceleration
- Hooke's law: $F = k(x - x_{\text{rest}})$
- Viscous friction: $F = c\, v$

$$
\boxed{\begin{aligned}
m_1\ddot{q}_1 &= k_2(q_2 - q_1) - k_1 q_1 \\
m_2\ddot{q}_2 &= k_3(u - q_2) - k_2(q_2 - q_1) - c\dot{q}_2
\end{aligned}}
$$

Converting models to state space form

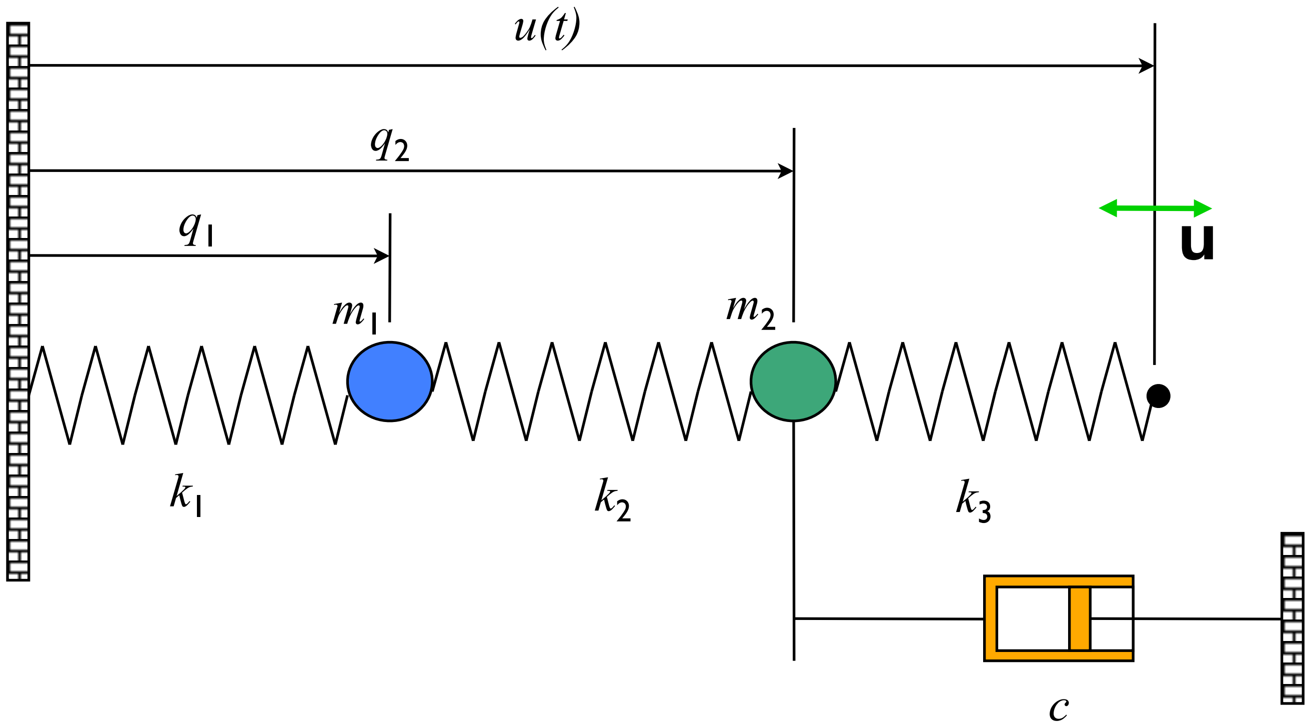
- Construct a *vector* of the variables that are required to specify the evolution of the system
- Write dynamics as a *system* of first order differential equations:

$$
\frac{d}{dt}\begin{bmatrix} q_1 \\ q_2 \\ \dot{q}_1 \\ \dot{q}_2 \end{bmatrix} = \begin{bmatrix} \dot{q}_1 \\ \dot{q}_2 \\ \frac{k_2}{m}(q_2 - q_1) - \frac{k_1}{m} q_1 \\ \frac{k_3}{m}(u - q_2) - \frac{k_2}{m}(q_2 - q_1) - \frac{c}{m}\dot{q} \end{bmatrix}
$$

$$
y = \begin{bmatrix} q_1 \\ q_2 \end{bmatrix}
$$

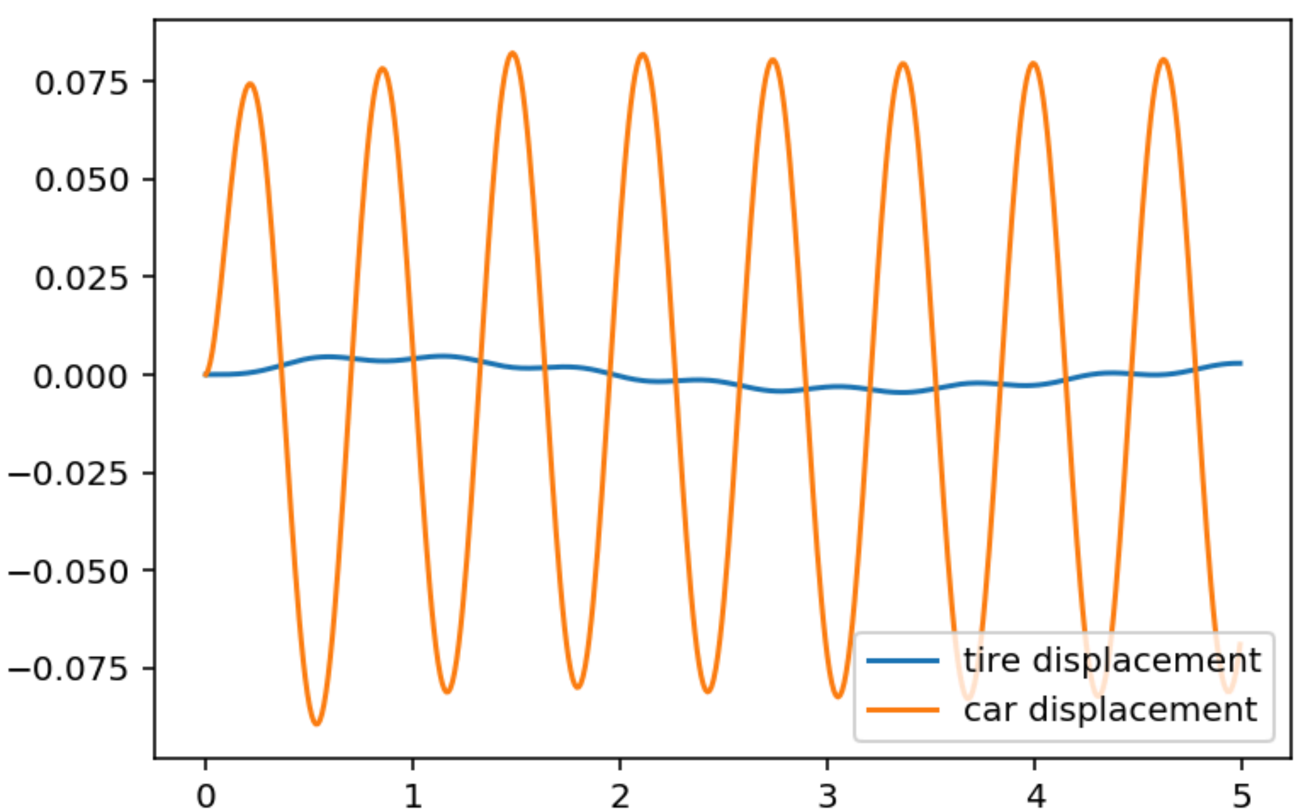
"State space form"

Simulating a state-space system



basic task: repeatedly calculate state update:

$$\boldsymbol{q}_{t+\Delta T} = \boldsymbol{q}_t + \Delta T \dot{\boldsymbol{q}}_t = \boldsymbol{q}_t + \Delta T \boldsymbol{f}(\boldsymbol{q})$$



Python simulation

```
import numpy as np
import matplotlib.pyplot as plt

k1 = k2 = k3 = m1 = c = 1
m2 = 0.1
dt = 0.01

time = np.arange(0, 5, dt)
q_data = np.zeros((len(time), 4))
q = np.array((0, 0, 0, 0))     ← initial condition
def f(q, u):                  ← dynamics function
     return np.array((
         q[2],
         q[3],
         -(k1+k2)/m1*q[0] + k2/m1*q[1],
         k2/m2*q[0] - (k2+k3)/
             m2*q[1] - c/m2*q[3] + k3/m2*u))

for idx, t in enumerate(time):
    u = np.cos(10*t)
    q = q + dt * f(q, u)          ← update step
    q_data[idx,:] = q             ← store result
plt.plot(time, q_data[:,0:2])
plt.legend(('car displacement (q1)',
            'tire displacement (q2)'))
```

general form of differential equations

State space form

$$\frac{dx}{dt} = f(x,u) \qquad y = h(x,u)$$

General form

$$\frac{dx}{dt} = Ax + Bu \qquad y = Cx + Du$$

Linear system

$$x \in \mathbb{R}^n, u \in \mathbb{R}^p \qquad y \in \mathbb{R}^q$$

• x = state; *n*th order

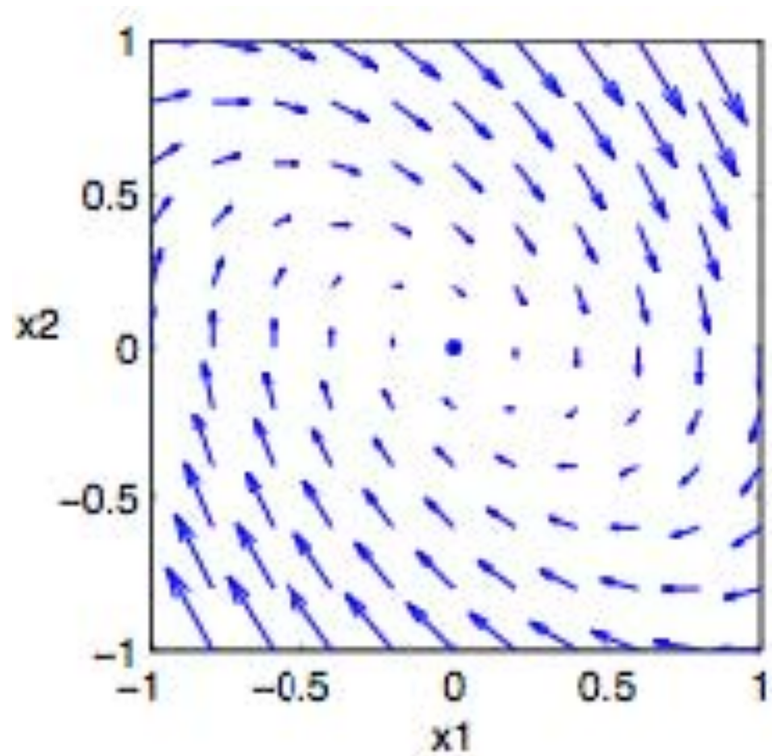
phase plots show 2D behavior

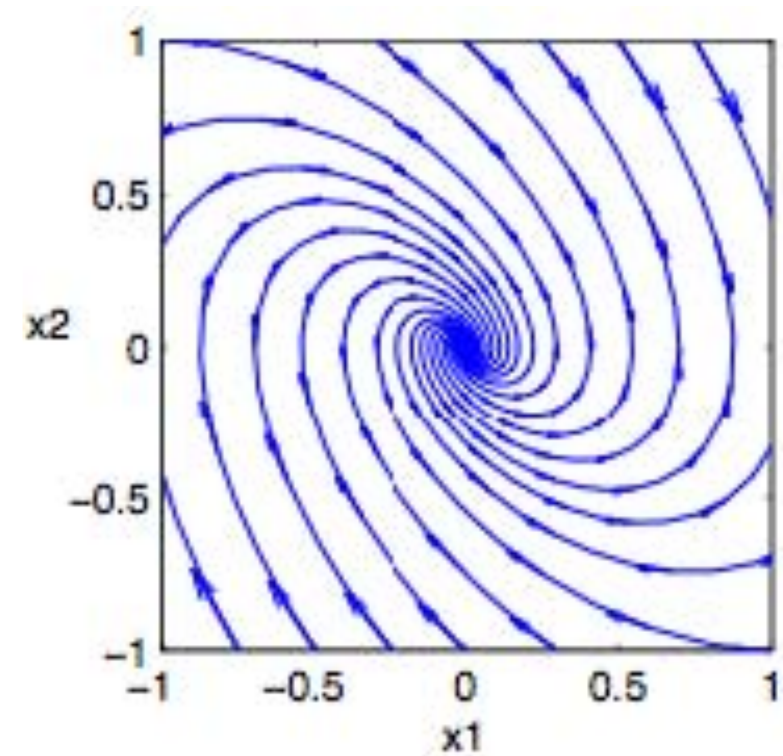
Phase plane plots show 2D dynamics as *vector fields* & *stream functions*

- $\dot{x} = f(x, u(x)) = F(x)$
- Plot *F*(*x*) as a vector on the plane; stream lines follow the flow of the arrows

$$\frac{d}{dt}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_2 \\ -x_1 - x_2 \end{bmatrix}$$

python: use 'streamplot' function in Matplotlib





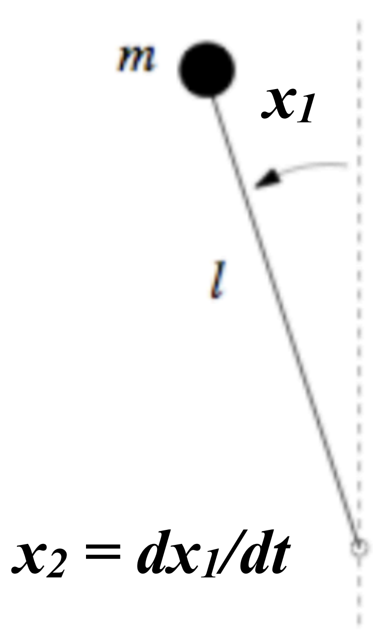
equilibrium points

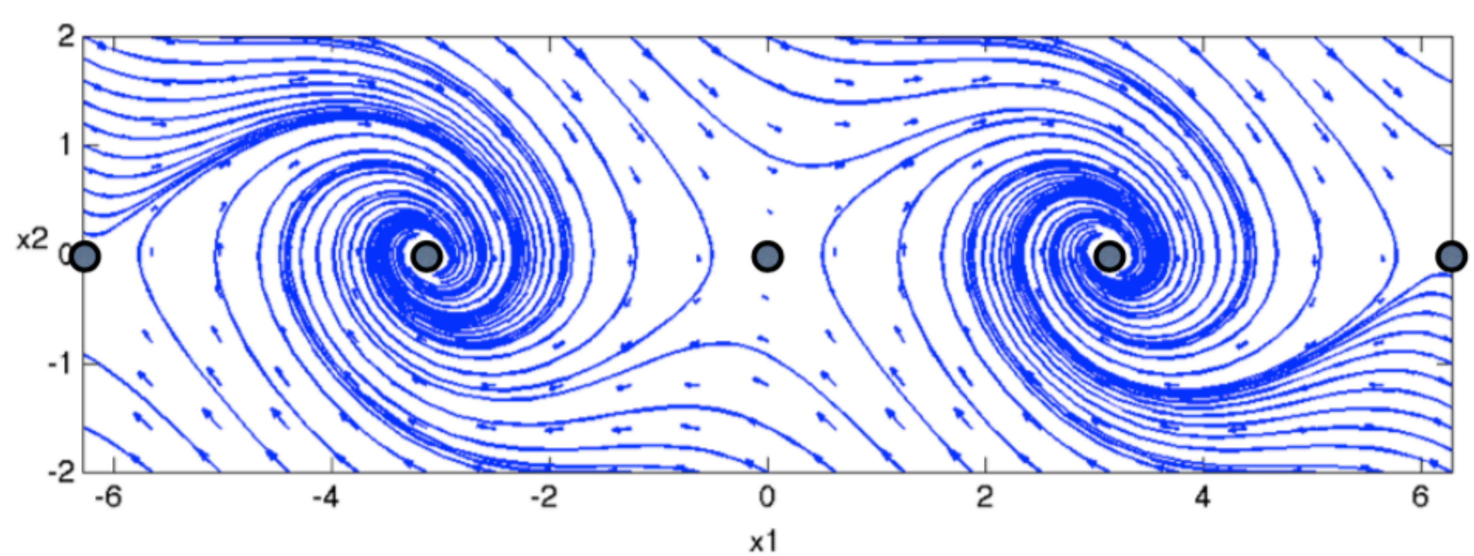
Equilibrium points represent stationary conditions for the dynamics

The *equilibria* of the system $\dot{x} = f(x)$ are the points x_e such that $f(x_e) = 0$.

Example:

$$\frac{dx}{dt} = \begin{bmatrix} x_2 \\ \sin x_1 - \gamma x_2 \end{bmatrix} \qquad \Rightarrow \qquad x_e = \begin{bmatrix} \pm n\pi \\ 0 \end{bmatrix}$$



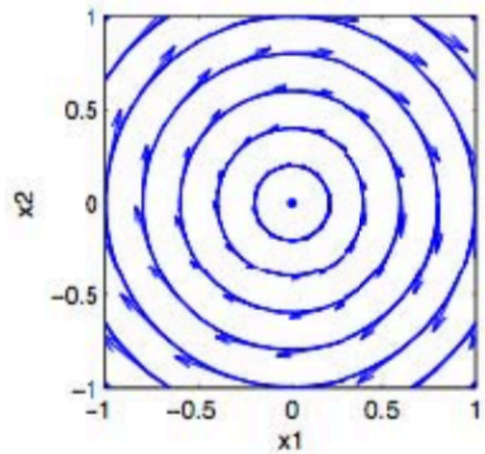


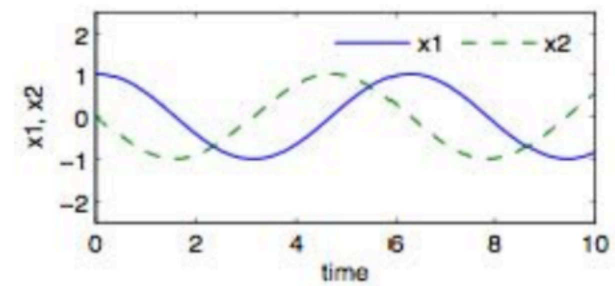
stability of equilibrium points

An equilibrium point is:

Stable if initial conditions that start near the equilibrium point, stay near

- Also called "stable in the sense of Lyapunov

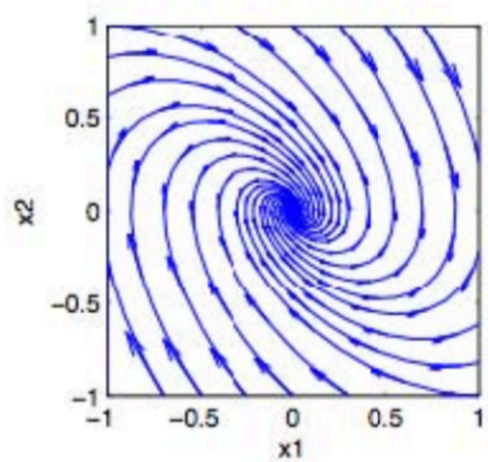


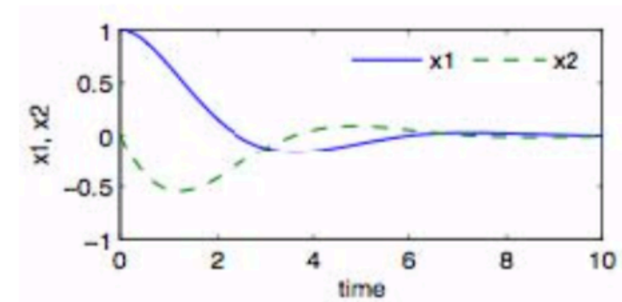


"stable" but not asymptotically stable

Asymptotically stable if all nearby initial conditions converge to the equilibrium point

- Stable + converging

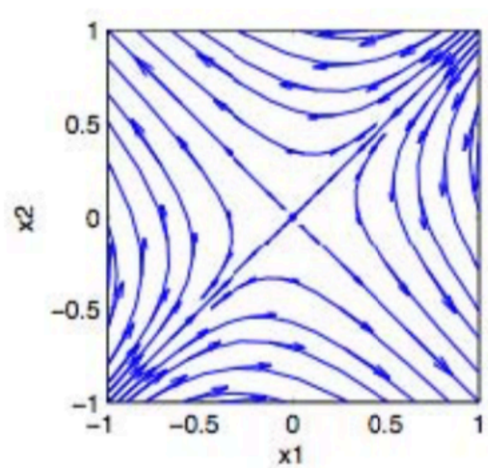


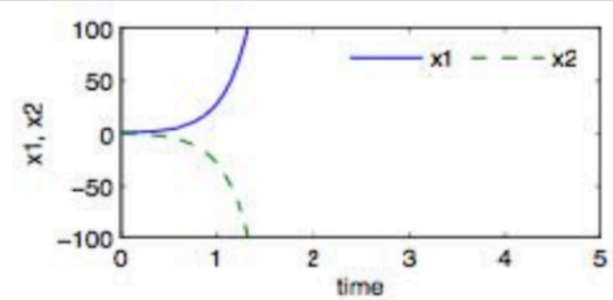


asymptotically stable

Unstable if some initial conditions diverge from the equilibrium point

- May still be some initial conditions that converge



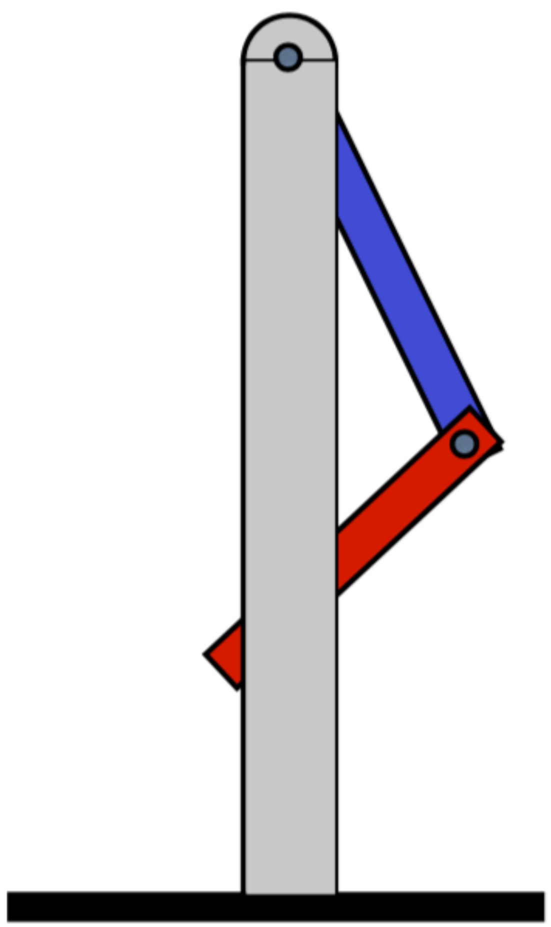


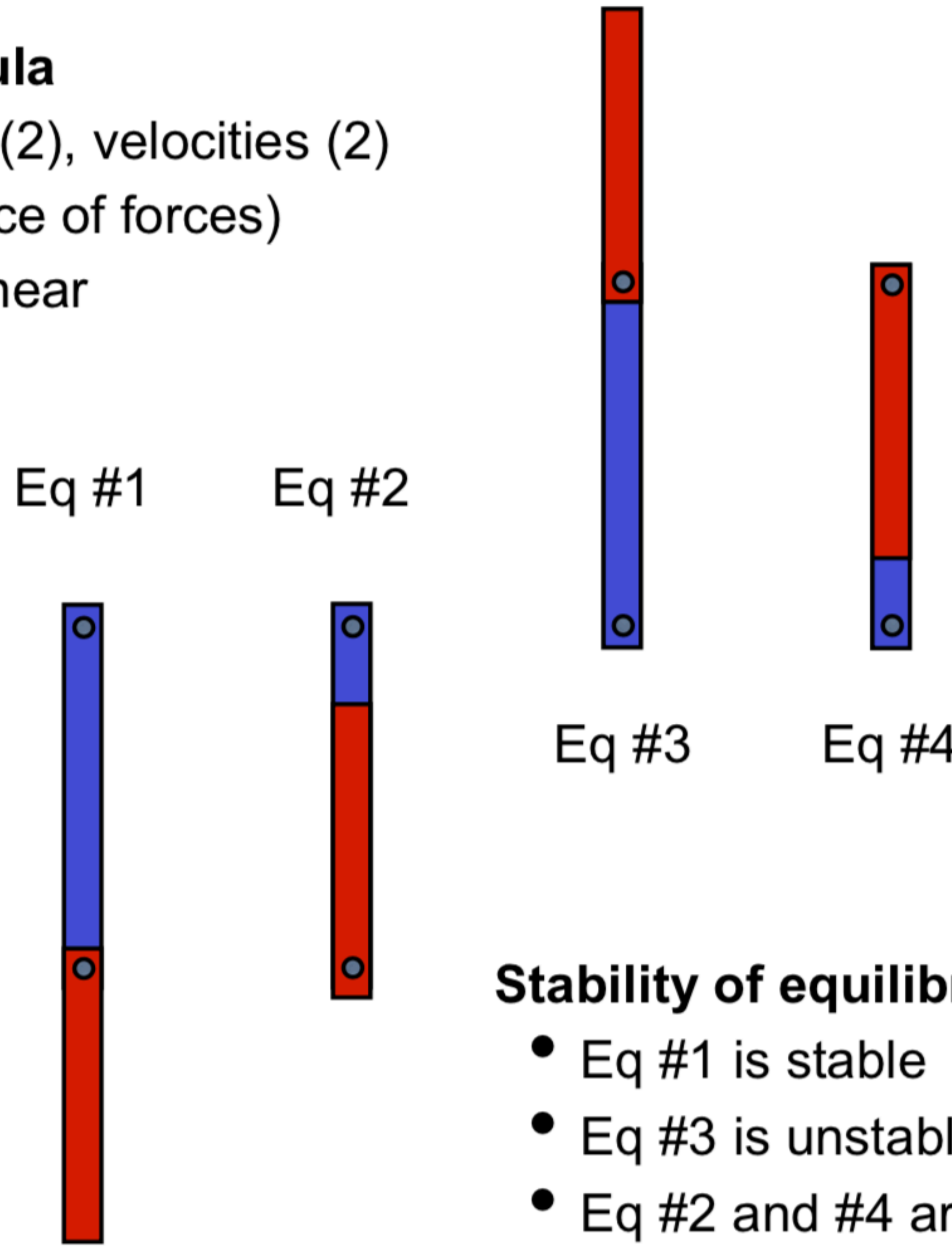
unstable

Example #1: Double Inverted Pendulum

Two series coupled pendula

- States: pendulum angles (2), velocities (2)
- Dynamics: $F = ma$ (balance of forces)
- Dynamics are very nonlinear





Stability of equilibria

- Eq #1 is stable
- Eq #3 is unstable
- Eq #2 and #4 are unstable, but with some stable “modes”

Stability of linear systems $\dot{x} = Ax$

- Theorem: linear system is asymptotically stable if and only if all eigenvalues λ of A have negative real part.

Local stability of nonlinear systems $\dot{x} = F(x)$

Asymptotic stability of the linearization implies *local* asymptotic stability of equilibrium point

- Linearization around equilibrium point captures "tangent" dynamics

$$\dot{x} = \overset{0}{\cancel{F(x_e)}} + \left.\frac{\partial F}{\partial x}\right|_{x_e} (x - x_e) + \text{higher order terms} \xrightarrow{\text{approx}} \begin{array}{l} z = x - x_e \\ \dot{z} = Az \end{array}$$

- linearization is *stable* $\Longrightarrow$ nonlinear system *locally stable*
- linearization is *unstable* $\Longrightarrow$ nonlinear system *locally unstable*

- "degenerate case": if linearization is *stable* but not *asymptotically stable* $\Longrightarrow$ cannot tell whether nonlinear system is stable or not!

$$\dot{x} = \pm x^3 \xrightarrow{\text{linearize}} \dot{x} = 0$$

- linearization is stable (but not asy stable)
- nonlinear system can be asy stable or unstable

Local linear approximation is valuable for control design:

- if dynamics are well-approximated by linearization near an equilibrium point, controller can *ensure* stability there (!)
- controller task: make the linearization stable

Linearization about an equilibrium point

$$\dot{x} = f(x,u) \qquad \longrightarrow \qquad \dot{z} = Az + Bv$$
$$y = h(x,u) \qquad \qquad \qquad w = Cz + Dv$$

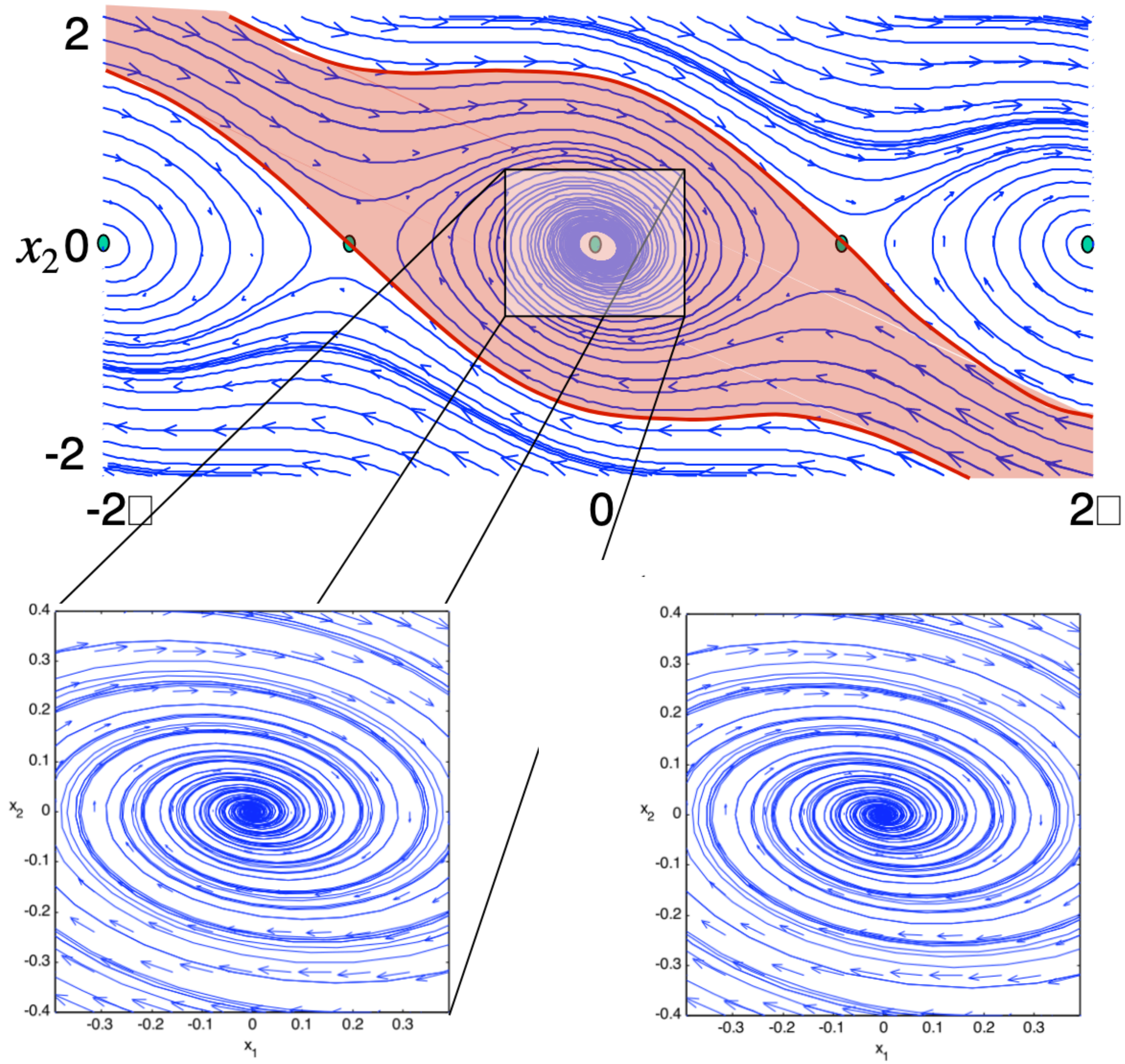
to "linearize" around $x = x_e$:

1. **find x_e, u_e such that $f = 0$**
2. **define** $y_e = h(x_e, u_e)$

   $z = x - x_e \qquad v = u - u_e \qquad w = y - y_e$
3. **then**

$$A = \left.\frac{\partial f}{\partial x}\right|_{(x_e,u_e)} \qquad B = \left.\frac{\partial f}{\partial u}\right|_{(x_e,u_e)}$$

$$C = \left.\frac{\partial h}{\partial x}\right|_{(x_e,u_e)} \qquad D = \left.\frac{\partial h}{\partial u}\right|_{(x_e,u_e)}$$



full nonlinear model **linear model (honest!)**

Remarks

- In examples, this is often equivalent to small angle approximations, etc
- Only works *near* to equilibrium point
- use linearization to design controller

big idea: if combined linearized system + controller is stable
$\Longrightarrow$ nonlinear system (incl control) is stable nearby

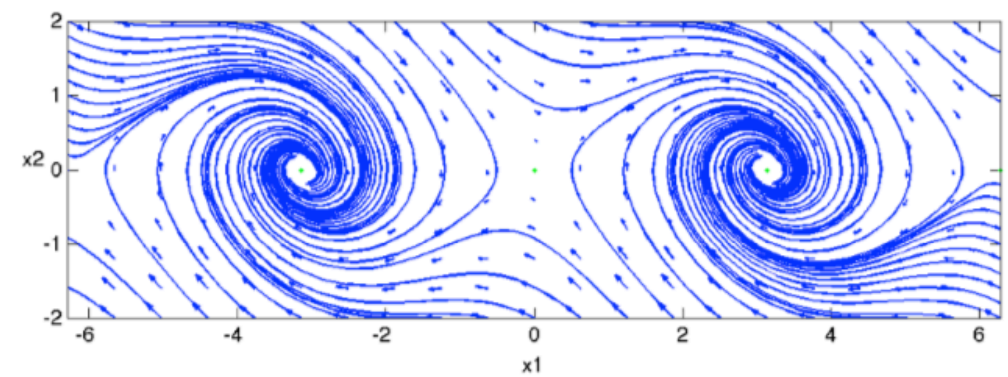
Jacobian linearization matrix

$$A = \left.\frac{\partial f}{\partial x}\right|_{(x_e, u_e)} = \left.\begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \cdots & \frac{\partial f_1}{\partial x_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial f_m}{\partial x_1} & \cdots & \frac{\partial f_m}{\partial x_n} \end{bmatrix}\right|_{(\boldsymbol{x_e}, \boldsymbol{u_e})}$$

Example: Stability Analysis of Inverted Pendulum

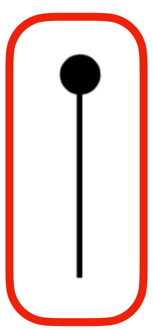
System dynamics

$$\frac{dx}{dt} = \begin{bmatrix} x_2 \\ \sin x_1 - \gamma x_2 \end{bmatrix},$$



Equilibria: where $\dot{x} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow \; x_e = \begin{bmatrix} \pm\pi k, \quad k = 0,\ 1,\ 2,\ 3\ldots \\ 0 \end{bmatrix}$

Linearize to assess stability: $\dfrac{\partial f}{\partial x} = \begin{bmatrix} \dfrac{\partial f_1}{\partial x_1} & \dfrac{\partial f_1}{\partial x_2} \\ \dfrac{\partial f_2}{\partial x_1} & \dfrac{\partial f_2}{\partial x_2} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ \cos x_1 & -\gamma \end{bmatrix}$

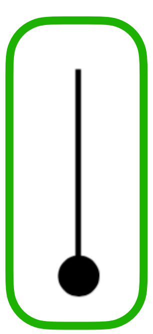


Upward equilibria: $x_1 = \pm 2\pi k, \quad k = 0,\ 1,\ 2,\ 3\ldots$

$$A = \frac{\partial f}{\partial x}\Big|_{x_e} = \begin{bmatrix} 0 & 1 \\ 1 & -\gamma \end{bmatrix}$$

eigenvalues: $\lambda = -\frac{1}{2}\gamma \pm \frac{1}{2}\sqrt{4+\gamma^2}$

for $\gamma = \mathbf{0.1},\ \lambda \cong (\mathbf{0.95},\ \mathbf{-1.05}) \Longrightarrow$ **unstable**



Downward equilibria: $x_1 = \pi \pm 2\pi k, \quad k = 0,\ 1,\ 2,\ 3\ldots$

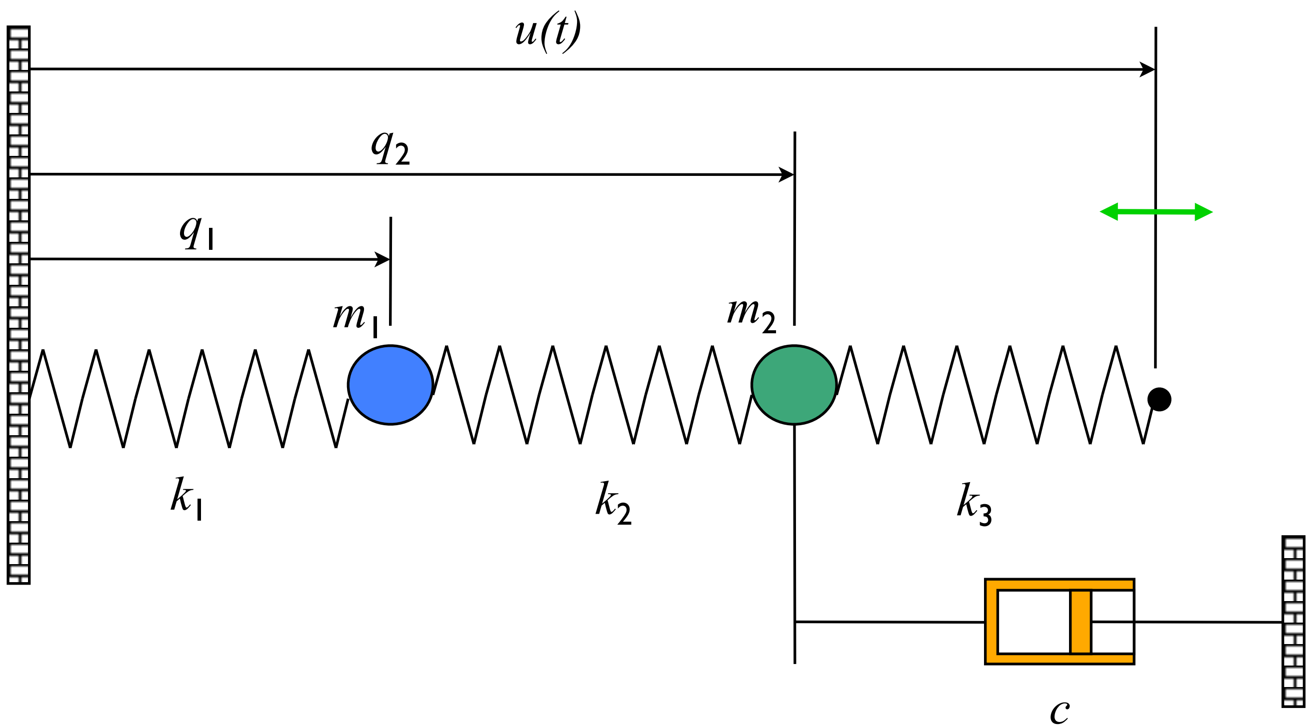
use $z_1 = x_1 - x_{1e} = x_1 - \pi, \quad z_2 = x_2 \quad \Rightarrow \dot{z} = Az$

$$A = \frac{\partial f}{\partial x}\Big|_{x_e} = \begin{bmatrix} 0 & 1 \\ -1 & -\gamma \end{bmatrix}$$

eigenvalues: $\lambda = -\frac{1}{2}\gamma \pm \frac{1}{2}\sqrt{-4+\gamma^2}$

for $\gamma = \mathbf{0.1},\ \lambda \cong (\mathbf{-0.05}+i,\ \mathbf{-0.05}-i) \Longrightarrow$ **stable**

example 2: matrix representation of a linear system



Model: rigid body physics

- Sum of forces = mass * acceleration
- Hooke's law: $F = k(x - x_{rest})$
- Viscous friction: $F = c\, v$

$$\boxed{\begin{aligned} m_1\ddot{q}_1 &= k_2(q_2 - q_1) - k_1 q_1 \\ m_2\ddot{q}_2 &= k_3(u - q_2) - k_2(q_2 - q_1) - c\dot{q}_2 \end{aligned}}$$

Matrix representation:

$$\dot{x} = Ax + Bu$$

$$\frac{d}{dt}\begin{bmatrix} q_1 \\ q_2 \\ \dot{q}_1 \\ \dot{q}_2 \end{bmatrix} = \begin{bmatrix} \dot{q}_1 \\ \dot{q}_2 \\ \frac{k_2}{m}(q_2 - q_1) - \frac{k_1}{m}q_1 \\ \frac{k_3}{m}(u - q_2) - \frac{k_2}{m}(q_2 - q_1) - \frac{c}{m}\dot{q} \end{bmatrix}$$

$$y = \begin{bmatrix} q_1 \\ q_2 \end{bmatrix}$$

"State space form"

$$\dot{x} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -\frac{k_1 + k_2}{m} & \frac{k_2}{m} & 0 & 0 \\ -\frac{k_2}{m} & -\frac{k_2 + k_3}{m} & 0 & 0 \end{bmatrix} x + \begin{bmatrix} 0 \\ 0 \\ 0 \\ \frac{k_3}{m} \end{bmatrix} u$$

$$y = [1 \quad 1 \quad 0 \quad 0]x = Cx$$

State Space Control Design Concepts

System description: single input, single output system (MIMO also OK)

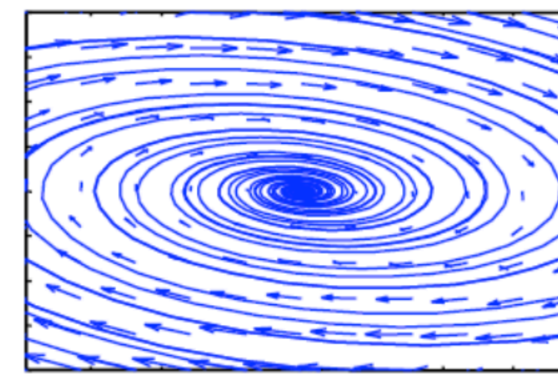
$$\begin{aligned} \dot{x} &= f(x,u) \qquad & x \in \mathbb{R}^n,\ x(0) \text{ given} \\ y &= h(x) \qquad & u \in \mathbb{R},\ y \in \mathbb{R} \end{aligned}$$

Stability: stabilize the system around an equilibrium point

- Given equilibrium point $x_e \in \mathbb{R}^n$, find control “law” $u = \alpha(x)$ such that

$$\lim_{t\to\infty} x(t) = x_e \text{ for all } x(0) \in \mathbb{R}^n$$

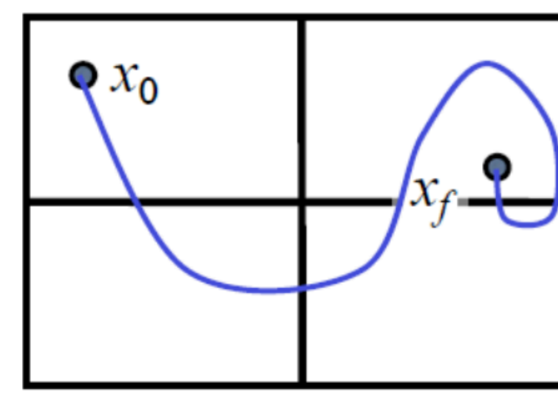
- Often choose x_e so that $y_e = h(x_e)$ has desired value r (constant)



Reachability: steer the system between two points

- Given $x_o, x_f \in \mathbb{R}^n$, find an input $u(t)$ such that

$$\dot{x} = f(x, u(t)) \text{ takes } x(t_0) = x_0 \rightarrow x(T) = x_f$$



Tests for Reachability

$$\dot{x} = Ax + Bu \qquad x \in \mathbb{R}^n,\ x(0) \text{ given}$$
$$y = Cx \qquad u \in \mathbb{R},\ y \in \mathbb{R}$$

$$x(T) = e^{AT}x_0 + \int_{\tau=0}^{T} e^{A(T-\tau)}Bu(\tau)d\tau$$

Thm A linear system is reachable if and only if the $n \times n$ *reachability matrix*

$$\begin{bmatrix} B & AB & A^2B & \cdots & A^{n-1}B \end{bmatrix}$$

is full rank.

Note: also called "controllability" matrix

Remarks

- *Very* simple test : `control.ctrb(A,B)` and check rank with `numpy.linalg.matrix_rank()`
- If this test is satisfied, we say "the pair (A,B) is reachable"

State space controller design for linear systems

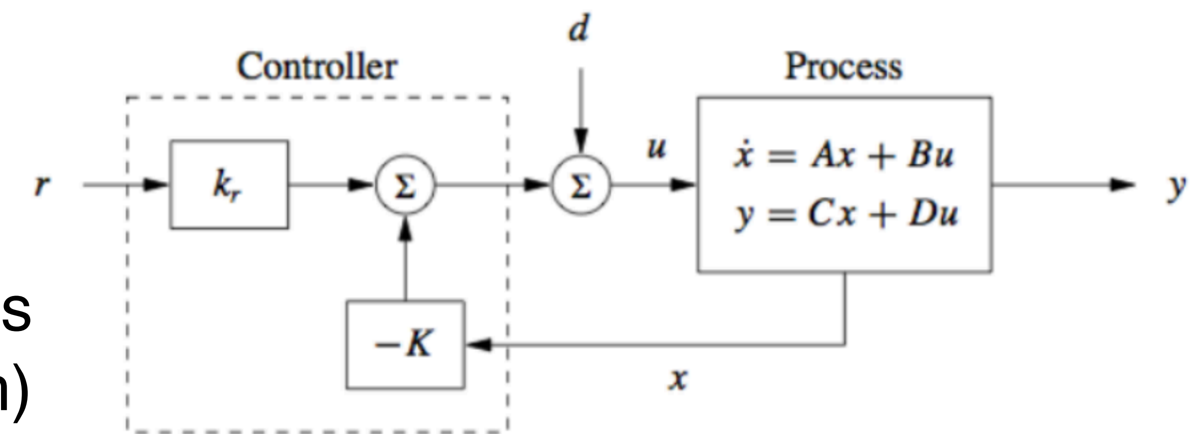
$$\dot{x} = Ax + Bu \qquad x \in \mathbb{R}^n,\ x(0) \text{ given}$$
$$y = Cx \qquad u \in \mathbb{R},\ y \in \mathbb{R}$$

$$x(T) = e^{AT}x_0 + \int_{\tau=0}^{T} e^{A(T-\tau)}Bu(\tau)d\tau$$

Goal: find a linear control law $u = -Kx$ such that the closed loop system

$$\dot{x} = Ax + Bu = (A - BK)x$$

is stable at $x = 0$ (assumes x are coordinates relative to location of equilibrium)



- Stability based on eigenvalues ⇒ use K to make eigenvalues of $(A - BK)$ stable
- Can also link eigenvalues to *performance* (eg, initial condition response)
- Question: when can we place the eigenvalues anyplace that we want?

Theorem The eigenvalues of $(A - BK)$ can be set to arbitrary values if and only if the pair (A, B) is reachable.

Next: one way to choose *K*