Exam procedures.

- Please write your name above.
- Please sit away from other students.
- If you have a question about the exam, please ask.
- This is a closed book exam. All relevant formulae are given on the exam. If you think a needed equation is not provided, please ask.
- Note that questions extend to the back of the sheets.
- Write your answers on the exam. I have tried to leave ample space, but if you need more, use additional paper and be sure to write “519 Midterm” and your name on the top of each page.
- Unless stated otherwise, it is necessary for full credit that you explain the logic of your calculation, deriving any results that you use. Standard results given in the equations sheet do not, however, require explanation.
1. **Perturbation Theory Toy Model (15 pts)**

Consider the Hermitian matrix

\[ M = \begin{pmatrix} A & B \\ B^* & A \end{pmatrix} \]

where \( A \) is real. Find the eigenvalues of \( M \) in 3 different ways. First, treat \( A \) as a small perturbation (that is, find the eigenvalues for \( A = 0 \) and then do first order perturbation theory in \( A \)), then treat \( B \) as a small perturbation and, last but not least, find the exact eigenvalues and compare to your two perturbative expansions.

- **For small \( A \)** we have \( M = M_0 + M_1 \) with

  \[ M_0 = \begin{pmatrix} 0 & B \\ B^* & 0 \end{pmatrix}, \quad M_1 = \begin{pmatrix} A & 0 \\ 0 & A \end{pmatrix}. \]

  The eigenvalues of \( M_0 \) are \( \pm |B| \). Since this spectrum is non-degenerate, we can use non-degenerate PT and read off the first correction simply from the diagonal entries in \( M_1 \). This is only true in the basis in which \( M_0 \) is diagonal, so we really should first diagonalize \( M_0 \). This can be accomplished by a unitary transformation

  \[ M_0 \rightarrow U M_0 U^{-1}, \quad M_1 \rightarrow U M_1 U^{-1}. \]

  \( U \) is made from the eigenvectors of \( M_0 \) and can easily be written down. However note that \( A \) is proportional to the identity so it does not change under this unitary transformation. To linear order in \( A \) the eigenvalues are

  \[ \lambda_1 = +|B| + A, \quad \lambda_2 = -|B| + A. \]

- **For small \( B \)** we have \( M = M_0 + M_1 \) with

  \[ M_0 = \begin{pmatrix} A & 0 \\ 0 & A \end{pmatrix}, \quad M_1 = \begin{pmatrix} 0 & B \\ B^* & 0 \end{pmatrix}. \]

  The only eigenvalue of \( M_0 \) is \( A \), but it is two-fold degenerate. Therefore we got to resort to degenerate PT. Our task is to diagonalize \( M_1 \) on the degenerate subspace. This is trivial to do and we find that the leading order corrections is given by the eigenvalues of \( M_1 \) which are \( \pm |B| \). To linear order in \( B \) the eigenvalues are

  \[ \lambda_1 = A + |B|, \quad \lambda_2 = A - |B|. \]

- **The exact eigenvalues can be obtained from**

  \[ 0 = \det \begin{pmatrix} A - \lambda & B \\ B^* & A - \lambda \end{pmatrix} = (A - \lambda)^2 - |B|^2. \]

  This fixes the exact eigenvalues to be

  \[ \lambda_1 = A + |B|, \quad \lambda_2 = A - |B|. \]

Both our perturbative expansions happened to give the exact answer at first order!

2. **Breaking Degeneracy. (20 pts)** Consider an electron in a spherically symmetric potential. The spatial part of the wavefunction of the electron in an eigenstate of the Hamiltonian with energy \( E_1 \) has the form

\[ \psi_1(\vec{r}) = x f(|\vec{r}|), \]

and satisfies the normalization condition

\[ \int d^3r \left| \psi_1(\vec{r}) \right|^2 = 1. \]

(Here \( x \) is a Cartesian component of the position vector \( \vec{r} \).)
3. **Time Dependence. (45 pts)**

The eigenstates of a rotationally symmetric 2d harmonic oscillator are given by

\[ E_{n_x,n_y} = \hbar \omega (n_x + n_y + 1) \]

where \( n_x \) and \( n_y \) are occupation numbers of simple 1d SHO’s in the \( x \) and \( y \) direction respectively. In particular, there are two states with energy \( E = 2\hbar \omega \): \( |n_x = 1, n_y = 0 \rangle \) and \( |n_x = 0, n_y = 1 \rangle \).

You may find the following formulas about the 1d SHO useful:

\[ x = \sqrt{\frac{\hbar}{2m\omega}} (a^\dagger + a), \quad p = i \sqrt{\frac{\hbar m \omega}{2}} (a^\dagger - a), \quad a^\dagger |n \rangle = \sqrt{n + 1}|n + 1 \rangle, \quad a|n \rangle = \sqrt{n}|n - 1 \rangle. \]

(a) **(5 pts)** At time \( t = 0 \) the system is prepared in the \( |n_x = 1, n_y = 0 \rangle \) eigenstate. Determine the probability \( P(t) \) that the system, at time \( t \), is found in the \( |n_x = 0, n_y = 1 \rangle \) eigenstate.

The initial state is an energy eigenstate and so only picks up a trivial phase under time evolution. Even though there is a second energy eigenstate with the same eigenvalue, there will be no transition: \( P(t) = 0 \).

(b) **(15 pts)** Very shortly after preparing the state (still \( t = 0 \)) the Hamiltonian of the system is perturbed by

\[ V_1 = \lambda xy. \]

with \( \lambda \) being small. Once more, determine the probability \( P(t) \) that the system, at time \( t \), is found in the \( |n_x = 0, n_y = 1 \rangle \) eigenstate. Work to leading non-trivial order in \( \lambda \) (the first non-vanishing term in \( P(t) \) that contains \( \lambda \) for very small \( \lambda \)).
Here we can apply time dependent perturbation theory. Calling the initial state \( |10\rangle \) and the final state \( |01\rangle \), we have for the transition probability

\[
P(t) = |\tilde{c}_{01,10}|^2 = -\frac{i}{\hbar} \int_0^t dt' e^{i\omega_{01,10}t} \langle 10|V_1|01 \rangle^2.
\]

Since the two states are degenerate we have

\[
\omega_{01,10} = (E_{10} - E_{01})/\hbar = 0
\]

and so the integrand becomes completely time independent and so the integral can easily be performed:

\[
P(t) = \frac{t^2}{\hbar^2} |\langle 01|V_1|10 \rangle|^2.
\]

The last remaining task is to calculate the matrix element

\[
V_{01,10} = \frac{\hbar \lambda}{2m\omega} \langle 0|a + a^\dagger|1 \rangle \langle 1|a + a^\dagger|0 \rangle = \frac{\hbar \lambda}{2m\omega}
\]

So we finally obtain

\[
P(t) = \frac{\lambda^2}{4m^2\omega^2} t^2.
\]

(c) (10 pts) In class we derived, for the case of a sudden perturbation just as the \( V_1 \) we added in part b) a relation we called “Fermi’s golden rule” for the transition rate \( R \):

\[
R = \rho(E_n)|V_{nm}|^2 \frac{2\pi}{\hbar}.
\]

Briefly explain why the answer to part b) doesn’t even qualitatively agree with this rule.

Fermi’s golden rule gives us a rate, so a probability growing linearly in time, to decay into a state with the same energy. This seems to be exactly the case we are looking at here, but we find \( P \) growing quadratically with \( t \) instead. The difference is because for Fermi’s golden rule we were considering decays into a continuum of states. We found that the range of energies around the initial energies for which we had a non-vanishing transition probability shrank as \( 1/t \), while the likelihood to decay into any of them grew as \( t^2 \), just as we found here. The probability to decay into any final state hence grew as \( t \). Here we are considering a single final state and the probability goes as \( t^2 \).

(d) (15 pts) Show that the potential, including the perturbation of part b), can be rewritten as

\[
V = \frac{m\omega_+^2}{2} x_+^2 + \frac{m\omega_-^2}{2} x_-^2 + \frac{\omega_+^2 + \omega_-^2}{4} + xym \frac{\omega_+^2 - \omega_-^2}{2} = \frac{m\omega^2}{2} (x^2 + y^2) + xy\lambda.
\]

Given the definitions of \( x_\pm \), we can calculate

\[
x_\pm^2 = \frac{x^2 + y^2}{2} \pm xy
\]

and so

\[
V = \frac{m\omega_+^2}{2} x_+^2 + \frac{m\omega_-^2}{2} x_-^2 = (x^2 + y^2)m \omega_+^2 + \omega_-^2 + xym \frac{\omega_+^2 - \omega_-^2}{2} = \frac{m\omega^2}{2} (x^2 + y^2) + xy\lambda.
\]
So if we write our system as SHOs in the \(x_\pm\) directions, the eigentstates labeled by \(n_\pm\) are eigenstates of the full Hamiltonian and so, as usual, only evolve with a phase. Given the definition of \(x_\pm\), we can similarly define creation operators

\[
a_\pm = \frac{1}{\sqrt{2}}(a_x \pm a_y).
\]

For the unperturbed SHO we can equally use the \(n_+\) and \(n_-\) labels for the eigenstates. From the form of the creation operators we see that we have

\[
|n_x = 1, n_y = 0\rangle = \frac{1}{\sqrt{2}}(|n_+ = 1, n_- = 0\rangle + |n_+ = 0, n_- = 1\rangle)
\]

and

\[
|n_x = 0, n_y = 1\rangle = \frac{1}{\sqrt{2}}(|n_+ = 1, n_- = 0\rangle - |n_+ = 0, n_- = 1\rangle).
\]

Since we know that at \(t = 0\) our state was \(|n_x = 1, n_y = 0\rangle\) we can write the full time dependent state as

\[
|\psi(t)\rangle = \frac{e^{i\alpha(t)}}{\sqrt{2}}(e^{-i\omega_- t}|n_+ = 1, n_- = 0\rangle + e^{-i\omega_+ t}|n_+ = 0, n_- = 1\rangle)
\]

where the common phase \(e^{i\alpha(t)}\) comes from the zero point energy common to both states. From this we can obtain \(P(t)\) by projecting:

\[
P(t) = |\langle n_x = 0, n_y = 1|\psi(t)\rangle|^2 = \frac{1}{4}|e^{-i\omega_+ t} - e^{-i\omega_- t}|^2 = \frac{1}{4}|1 - e^{-i\Delta\omega t}|^2 = \sin^2\left(\frac{\Delta\omega t}{2}\right)
\]

where

\[
\Delta\omega = \omega_+ - \omega_-.
\]

For small \(\lambda\) we can expand

\[
\Delta\omega = \sqrt{\omega^2 + \frac{\lambda}{m}} - \sqrt{\omega^2 - \frac{\lambda}{m}} \approx \omega \left(1 + \frac{\lambda}{2m\omega^2}\right) - \omega \left(1 - \frac{\lambda}{2m\omega^2}\right) + \ldots = \frac{\lambda}{m\omega} + \ldots
\]

and so obtain

\[
P(t) = \frac{\lambda^2 t^2}{4m^2\omega^2} + \ldots
\]

in perfect agreement with the perturbative answer.