Exam procedures.

• Please write your name above.
• Please sit away from other students.
• If you have a question about the exam, please ask.
• This is a closed book exam. All relevant formulae are given on the exam. If you think a needed equation is not provided, please ask.
• Note that questions extend to the back of the sheets.
• Write your answers on the exam. I have tried to leave ample space, but if you need more, use additional paper and be sure to write “519 Midterm” and your name on the top of each page.
• Unless stated otherwise, it is necessary for full credit that you explain the logic of your calculation, deriving any results that you use. Standard results given in the equations sheet do not, however, require explanation.
1. *Perturbation Theory Toy Model* (15 pts)

Consider the Hermitian matrix

\[ M = \begin{pmatrix} A & B \\ B^* & A \end{pmatrix} \]

where \( A \) is real. Find the eigenvalues of \( M \) in 3 different ways. First, treat \( A \) as a small perturbation (that is, find the eigenvalues for \( A = 0 \) and then do first order perturbation theory in \( A \)), then treat \( B \) as a small perturbation and, last but not least, find the exact eigenvalues and compare to your two perturbative expansions.
Extra space for problem 1
2. *Breaking Degeneracy.* (20 pts) Consider an electron in a spherically symmetric potential. The spatial part of the wavefunction of the electron in an eigenstate of the Hamiltonian with energy $E_1$ has the form

$$\psi_1(\vec{r}) = x f(|\vec{r}|),$$

and satisfies the normalization condition

$$\int d^3r |\psi_1(\vec{r})|^2 = 1.$$

(Here $x$ is a Cartesian component of the position vector $\vec{r}$.)

(a) (10 pts) Write down a complete set of linearly-independent wavefunctions that have the same energy eigenvalue $E_1$ and are related by rotational symmetry.
(b) (10 pts) A spin-orbit interaction of the form

$$V_{SO} = U_0 \vec{S} \cdot \vec{L}/\hbar^2,$$

perturbs the system. [$U_0$ is independent of $\vec{r}$, and $\vec{L}$ and $\vec{S}$ are the orbital and spin angular momentum operators, respectively.] For both $j = \ell \pm 1/2$, find the first order energy splitting of the $E_1$ level. $j$ here denotes a quantum number labeling the eigenvectors of total angular momentum $\vec{J} = \vec{L} + \vec{S}$. 
3. **Time Dependence. (45 pts)**

The eigenstates of a rotationally symmetric 2d harmonic oscillator are given by

\[ E_{n_x,n_y} = \hbar \omega (n_x + n_y + 1) \]

where \( n_x \) and \( n_y \) are occupation numbers of simple 1d SHO’s in the \( x \) and \( y \) direction respectively. In particular, there are two states with energy \( E = 2\hbar \omega \): \( |n_x = 1, n_y = 0 \rangle \) and \( |n_x = 0, n_y = 1 \rangle \).

You may find the following formulas about the 1d SHO useful:

\[ x = \sqrt{\frac{\hbar}{2m \omega}} (a^\dagger + a), \quad p = i \sqrt{\frac{\hbar m \omega}{2}} (a^\dagger - a), \quad a^\dagger |n \rangle = \sqrt{n+1} |n+1 \rangle, \quad a |n \rangle = \sqrt{n} |n-1 \rangle. \]

(a) **(5 pts)** At time \( t = 0 \) the system is prepared in the \( |n_x = 1, n_y = 0 \rangle \) eigenstate. Determine the probability \( P(t) \) that the system, at time \( t \), is found in the \( |n_x = 0, n_y = 1 \rangle \) eigenstate.
(b) (15 pts) Very shortly after preparing the state (still $t = 0$) the Hamiltonian of the system is perturbed by

$$V_1 = \lambda xy.$$ 

with $\lambda$ being small. Once more, determine the probability $P(t)$ that the system, at time $t$, is found in the $|n_x = 0, n_y = 1\rangle$ eigenstate. Work to leading non-trivial order in $\lambda$ (the first non-vanishing term in $P(t)$ that contains $\lambda$ for very small $\lambda$).
(c) (10 pts) In class we derived, for the case of a sudden perturbation just as the $V_1$ we added in part b) a relation we called “Fermi’s golden rule” for the transition rate $R$:

$$R = \rho(E_n)|V_{nm}|^2 \frac{2\pi}{\hbar}.$$ 

Briefly explain why the answer to part b) doesn’t even qualitatively agree with this rule.
(d) (15 pts) Show that the potential, including the perturbation of part b), can be rewritten as

\[ V = \frac{m\omega^2_+}{2} x^2_+ + \frac{m\omega^2_-}{2} x^2_- \]

with

\[ x_\pm = \frac{x \pm y}{\sqrt{2}}, \quad \omega^2_\pm = \omega^2 \pm \frac{\lambda}{m}. \]

Use this information to find \( P(t) \) defined in part b) exactly and compare to your perturbative answer in b).
Extra Space: