1. **Transmission through glass** Consider light traveling from $x = -\infty$, incident normally on a plate of glass of thickness $a$. The plate is parallel to the $yz$ plane, with one face at $x = 0$ and the other at $x = a$. The index of refraction is unity for $x < 0$ and $x > a$, and $n = 1.5$ for $0 \leq x \leq a$. The electromagnetic field in the region $x < 0$ is a superposition of right and left traveling waves which are the incident and reflected waves. In the region $0 \leq x \leq a$ there are both right and left traveling waves, and in the region $x > a$ there is only the transmitted right traveling wave.

   (a) Write $\mathbf{E}(x, t)$ and $\mathbf{B}(x, t)$ in the three regions, letting the polarization of the incident wave be in the $y$ direction. Write the four boundary conditions on the wave amplitudes.

   (b) Determine the transmission coefficient $T$, the ratio of transmitted intensity to incident intensity.

   (c) Plot $T$ as a function of $ka$, where $k$ is the incident wave number.

2. **Pulse of radiation moving in the $x$ direction.** Consider the fields $E(r, t) = \hat{i}F_1(x - ct) + \hat{j}F_2(x - ct) + \hat{k}F_3(x - ct)$, $cB(r, t) = \hat{i}G_1(x - ct) + \hat{j}G_2(x - ct) + \hat{k}G_3(x - ct)$, where the functions $F_i$, $G_i$ approach 0 in the limits $x \to \pm\infty$. These fields satisfy the wave equation and correspond to a pulse of radiation moving in the $+x$ direction. The Maxwell equations place severe restrictions on the components $F_i$, $G_i$.

   (a) Show that the Maxwell equations require that $F_1 = G_1 = 0$, $G_3 = F_2$ and $G_2 = -F_3$.

   (b) Suppose $F_2(\xi) = G_3(\xi) = E_0 \exp(-\xi^2/\alpha^2)$ and other components are 0. Make a sketch that shows a snapshot of the electric and magnetic fields at a time $t$.

3. **Polarization perpendicular to the plane of incidence** A plane electromagnetic wave is incident from one linear medium to another.

   (a) Impose the boundary conditions and obtain Fresnel equations for $\tilde{E}_{0R}$ and $\tilde{E}_{0T}$.

   (b) Compute the fraction of incident energy that is reflected.

   (c) For incident light polarized in a direction parallel to the plane of incidence the fraction of incident energy that is reflected is denoted $R_\parallel$. For perpendicular polarization the fraction reflected is denoted: $R_\perp$. Suppose the incident medium is air $n = 1$ and the second medium has an index of refraction close to 1. Furthermore the angle of incidence is nearly grazing $\theta_i \approx \pi/2$. Show that $R_\parallel \approx R_\perp$ and derive each to first-order in small quantities.

4. **Applying Maxwell’s Equations**

   (a) Can $\mathbf{B} = B_0 \cos ax \hat{i}$ be a solution of Maxwell’s equations? Explain your answer.

   (b) Consider a region of free space that contains no free charges and currents. Can $\mathbf{H} = H_0 \cos(\beta z) \hat{j}$ be a solution of Maxwell equations? The answer is yes or no, but to get credit you must use Maxwell’s equations to explain your answer.

   (c) A capacitor with circular parallel plates, with radius $a$ and separation $d \ll a$, has potential difference $V(t)$. Determine the magnitude of the magnetic field $B$ on the midplane of the capacitor at a distance $s > a$ from the symmetry axis.

   (d) Suppose the material between the capacitor plates was a medium with non-zero conductivity $\sigma$. Explain how your answer to part (c) would be be changed.
Region 1

1. Incident wave
\[ \mathbf{E}_I = E_0 \hat{y} e^{i(kx - wt)} \]
\[ \mathbf{B}_I = \frac{E_0}{c} \hat{z} e^{i(kx - wt)} \]
\[ \mathbf{E}_r = \mathbf{E}_R \hat{y} e^{i(-kx - wt)} \]
\[ \mathbf{B}_r = -\mathbf{E}_R \hat{z} e^{i(-kx - wt)} \]

Region 2

2. Here wave number is \( k_2 = n \omega / c \), \( n = 3/2 \)
\[ \mathbf{E}_{2R} = \mathbf{E}_{2R} \hat{y} e^{i(k_2 x - wt)} \]
\[ \mathbf{B}_{2R} = \frac{n \mathbf{E}_{2R}}{c} \hat{z} e^{i(k_2 x - wt)} \]
\[ \mathbf{E}_{2L} = \mathbf{E}_{2L} \hat{y} e^{i(-k_2 x - wt)} \]
\[ \mathbf{B}_{2L} = -\frac{n \mathbf{E}_{2L}}{c} \hat{z} e^{i(-k_2 x - wt)} \]

Region 3

\[ \mathbf{E}_3 = \mathbf{E}_3 \hat{y} e^{i(kx - wt)} \]
\[ \mathbf{B}_3 = \frac{\mathbf{E}_3}{c} \hat{z} e^{i(kx - wt)} \]

b) The boundary conditions at \( x = 0 \) and \( x = a \) are that tangential components of \( \mathbf{E} \) and \( \mathbf{B} = \mathbf{B}/\mu_0 \) are continuous (Glass is non magnetic)

This gives a total of 4 BC
which are needed. In the equation there are 4 unknowns: \( \tilde{E}_0 \), \( \tilde{E}_{2R} \), \( \tilde{E}_{2L} \), \( \tilde{E}_3 \), each of these is proportional to \( \tilde{E}_0 \) and the goal is to get \( \tilde{E}_3 \) because

\[
T = \left| \frac{\tilde{E}_{13}}{\tilde{E}_{13}^0} \right|^2
\]

at \( x = 0 \), tangential \( \tilde{E} \) continuous:

\[
\tilde{E}_0 + \tilde{E}_n = \tilde{E}_{2R} + \tilde{E}_{2L}
\]

1.

tangential \( \tilde{H} \):

\[
\tilde{H}_0 - \tilde{H}_n = n(\tilde{E}_{2R} - \tilde{E}_{2L})
\]

2.

at \( x = a \),

tangential \( \tilde{E} \) gives:

\[
\tilde{E}_{2R} e^{i \alpha a} + \tilde{E}_{2L} e^{-i \alpha a} = \tilde{E}_3 e^{i \alpha a}
\]

3. tangential \( \tilde{H} \) gives:

\[
n(\tilde{E}_{2R} e^{i \alpha a} - \tilde{E}_{2L} e^{-i \alpha a}) = \tilde{H}_3 e^{i \alpha a}
\]

4.

Eqs 1-4 represent 4 cases, 4 unknowns.

3, 4 \( \Rightarrow \)

\[
\tilde{E}_{2R} e^{i \alpha a} + \tilde{E}_{2L} e^{-i \alpha a} = n(\tilde{E}_{2R} e^{i \alpha a} - \tilde{E}_{2L} e^{-i \alpha a})
\]

\( \Rightarrow \)

\[
\tilde{E}_{2R} = \tilde{E}_{2L} \frac{(n-1)}{n+1} e^{i \alpha a}
\]

5 [if \( n = 1 \), \( \tilde{E}_{2L} = 0 \)]

\( 3 + 4 N \Rightarrow \)

\[
2 \tilde{E}_{2R} e^{i \alpha a} = \tilde{E}_3 e^{i \alpha a} \frac{n+1}{n}
\]

6
\[ 2 \tilde{E}_n = \tilde{E}_{2n} (1 + n) + \tilde{E}_{2n} (1 - n) \]

Use 5 in 7

\[ 2 \tilde{E}_n = \tilde{E}_{2n} (1 + n) + \frac{(n-1)^2 e^{-2i \alpha n}}{n+1} \]

\[ \tilde{E}_{2n} = \frac{2 \tilde{E}_n (n+1)}{(n+1)^2 - (n-1)^2 e^{-2i \alpha n}} \]

Use 6 in 8

\[ \tilde{E}_3 = \frac{2 \tilde{E}_n}{(n+1)^2 - (n-1)^2 e^{-2i \alpha n}} \]

\[ T = \left| \frac{\tilde{E}_3}{\tilde{E}_n} \right|^2 = \frac{16 n^2}{(n+1)^4 + (n-1)^4 - 2 (n^2 - 1) \cos 2 \alpha n} \]

\[ (n+1)^4 + (n-1)^4 - 2 (n^2 - 1)^2 = 16 n^2 \quad \text{so} \]

\[ T = \frac{16 n^2}{16 n^2 - 2 (n^2 - 1) \cos 2 \alpha n - 1} \]

\[ = \frac{16 n^2}{16 n^2 + 2 (n^2 - 1) \sin^2 \alpha n} \quad \text{use } \eta = \frac{1}{2} \]

\[ T = \frac{1}{1 + \frac{25}{288} \sin^2 \alpha n} \]

If \( n = 1 \) then \( T = 1 \)

If \( \alpha = 0 \) then \( T = 1 \)

(2 checks)

c) \( T \) varies between 1 and \( \sqrt{1 + \frac{25}{288}} = 0.92 \)

\[ T \]
2. a) We need $\nabla \cdot \mathbf{E}$ and $\nabla \times \mathbf{E}$
when acting on a function that depends on $x-ct$
these become $\frac{\partial}{\partial x}$ and $\frac{\partial}{\partial x}$

$\nabla \cdot \mathbf{E} = 0$ gives $\frac{\partial F_1}{\partial x} = 0 = F'_1(x-ct)$

So $F'_1 = 0$ integration gives a constant of integration
but the constant must be 0 because $\lim_{x \to 0} F'_1(x-ct) = 0$

Similarly $\nabla \times \mathbf{E} = 0$ gives $G_1 = 0$

Let's look at $\nabla \times \mathbf{B} = \mu_0 \varepsilon_0 \frac{\partial \mathbf{E}}{\partial t}$

$\nabla \times \mathbf{B} = \frac{1}{2} \frac{d}{dx} \left[ J G_2 + \mathbf{A} \cdot \mathbf{G}_3 \right]$

$= \frac{1}{2} \left[ \frac{\partial}{\partial t} (G_2') - \mathbf{A} \cdot \mathbf{G}_3' \right] \frac{\partial F_2}{\partial t}$

$c \mu_0 \varepsilon_0 \frac{\partial E}{\partial t} = \frac{-5}{c^2} \left[ \int F_2' + \mathbf{A} \cdot F_3' \right]$

Equate y-components
$\int F_2 = -G_2$, $\int F_3' = G_3'$

Integration gives
$F_2 = -B_2 + K_1$, $F_3 = G_2 + K_1$

but $K_1, K_3 = 0$ since $F_1, G_1, B_2 \to 0$ as $x \to 0$

b)
3) Start with boundary conditions:

If an arbitrary array or surface normal components of $E, B$ are tangential components of $E_T, B_T$ are continuous across the boundary.

Polarization is the change of medium:

$$\vec{E}_p = \vec{E}_j \times \vec{n}$$

Then:

$$\vec{E} = \vec{E}_j + \vec{E}_p$$

That is direction of $\vec{E}$ is in direction of $\vec{E}_j$ + direction of $\vec{E}_p$.

Now apply boundary conditions (Continuity of Electric Field) for:

Normal $\mathbf{E}$

$$E_n = E_{n1} + E_{n2} \Rightarrow 0 = 0$$

Tangential $\mathbf{E}$ and $\mathbf{B}$ have same sign.

Tangential $\mathbf{E}$

$$\frac{\partial E_b}{\partial n} = \frac{\partial B}{\partial n}$$

Thus:

$$\frac{\partial E_b}{\partial n} = \frac{\mu_0}{\epsilon_0} \left[ \frac{1}{\mu_0} \frac{\partial B}{\partial n} - \frac{1}{\epsilon_0} \frac{\partial E_b}{\partial n} \right]$$

$$\Rightarrow \frac{\partial E_b}{\partial n} = \frac{\mu_0}{\epsilon_0} \mu_0 \cos \theta = \frac{\partial E_b}{\partial n}$$

The hard equations are now equations without $\mathbf{E}$, $\mathbf{E}$ in normal $\mathbf{E}$.

Let $x = \cos \theta, p = \sin \theta$

adding 2: $\cos \theta \frac{\partial E_b}{\partial x} = \frac{\partial E_b}{\partial n}$
Given $E_{o2}$ use (b) to get $E_{o1}$

\[
E_{o1} = \frac{1-dg}{1+dg}
\]

$d, g$ are both positive.

The two boxed equations are by Fresnel.

Note $E_{o1}, E_{o2}, E_{r}$ are all complex numbers.

since $d \beta > 0 \quad E_{r}$ has the same phase as $E_{o2}$

\[d \beta > 0 \quad E_{r} \quad \text{has the same phase as } E_{o2}\]

\[\frac{E_{o1}}{E_{o2}} = \left| \frac{1-dg}{1+dg} \right| E_{o2}\]

(b) The reflected intensity $= \frac{1}{2} \cdot 6.07 \cdot |E_{o2}|^2$.

\[
\beta = \frac{J_{r}}{J_{o}} = \left( \frac{1-dg}{1+dg} \right)^2 = \beta_1
\]

(c) now $x = \frac{\cos \delta}{\cos \delta_1}$

\[\beta = \frac{x \pm \sqrt{x^2 - 1}}{2}\]

\[\beta_1 = \frac{x_1 \pm \sqrt{x_1^2 - 1}}{2}\]

where $\delta$ is small.

\[\delta = \frac{\pi}{2} \quad \delta_1 = \frac{\pi}{2}\]

\[\cos \delta = (1-e) \sin \delta_1 = (1-e) \sin \delta_1\]

\[\sin \left( \frac{\pi}{2} - \delta \right) = (1-e) \sin \delta_1\]

\[\Theta = \cos \delta = \left. \frac{\cos \delta}{\cos \delta_1} \right| \text{in question}\]

\[\Theta = \frac{\cos \delta}{\cos \delta_1} = \frac{1 - x^2}{2} = \frac{1 - (1 - \frac{1}{2} \sin^2 \delta)}{2}
\]

\[\frac{1 - x^2}{2} = E = \Theta_{1 - \frac{1}{2} \sin^2 \delta}\]

This justifies $\delta = \Theta_1$

\[\beta = 1 + e
\]

From (a) $R = \left( \frac{x - \beta}{x + \beta} \right)^2$

\[= \left( \frac{\cos \delta - \beta}{\cos \delta + \beta} \right)^2
\]

From previous page $R = \left( \frac{1 - \frac{1}{2} \sin^2 \delta}{1 + \frac{1}{2} \sin^2 \delta} \right)^2$
Multiply previous \( R \) by \( \frac{\delta^2}{\delta \sigma^2} \)

\[
R_l = \left( \frac{\delta / \sigma - \beta}{\delta / \sigma + \beta} \right)^2
\]

\( R_{11} = R_l \)

These are the same if

\( \delta \approx \theta \)

This is true if \( \epsilon \) is small enough because from previous page

\( y^2 = \delta^2 + 2 \epsilon \)

\( \epsilon \) must be of order \( \delta^3 \) or higher power
4. Applying Maxwell’s Equations

(a) Can \( \mathbf{B} = B_0 \cos ax \mathbf{i} \) be a solution of Maxwell’s equations? Explain your answer. \( \nabla \cdot \mathbf{B} = -B_0 a \sin ax \neq 0 \). This violates \( \nabla \cdot \mathbf{B} = 0 \), so the answer is \[ \text{no}. \]

(b) Consider a region of free space that contains no free charges and currents. Can \( \mathbf{H} = H_0 \cos(\beta z) \mathbf{j} \) be a solution of Maxwell equations? The answer is yes or no, but to get credit you must use Maxwell’s equations to explain your answer. The curl of \( \mathbf{H} \) is non-zero and is a vector in the \( x \)-direction that is a function of \( z \) and time-independent. Then equation (iv) of prob 2 says there is a non-zero time derivative of \( \mathbf{E} \) in the \( x \)-direction. This means that \( \mathbf{E} \) is a time dependent quantity in the \( x \)-direction that depends on \( z \). This \( \mathbf{E} \) has a time-dependent curl in the \( y \) direction. Then using equation (iii) of the previous problem gives an \( \mathbf{H} = \mathbf{B}/\mu_0 \) that is time-dependent. This is in conflict with the given time independence of \( \mathbf{H} \). Thus the answer is \[ \text{no}. \] You can also simply calculate the various terms.

(c) A capacitor with circular parallel plates, with radius \( a \) and separation \( d \ll a \), has potential difference \( V(t) \). Determine the magnitude of the magnetic field \( B \) on the midplane of the capacitor at a distance \( s > a \) from the symmetry axis.

\[
V(z,t) = -V(t) \frac{z}{d}, \quad \mathbf{E} = -\nabla V = \frac{V(t)}{d} \mathbf{k}.
\]
The displacement current density \( \mathbf{J}_D = \mu_0 \varepsilon_0 \frac{\partial \mathbf{E}}{\partial t} = \mu_0 \varepsilon_0 \frac{dV}{dt} \frac{1}{a} \)

Ampere’s law with loop of radius \( s \) to get \( 2\pi s B = \mu_0 \varepsilon_0 \frac{dV}{dt} \frac{1}{2} \pi a^2 \). (The enclosed current is through the area \( \pi a^2 \).) Solving gives \[ B = \frac{\mu_0 \varepsilon_0}{2s} \frac{dV}{dt} \frac{a^2}{2}. \]

(d) Suppose the material between the capacitor plates was a medium with non-zero conductivity \( \sigma \). Explain how your answer to part (c) would be be changed.

One must now include the current given by \( \mathbf{J} = \sigma \mathbf{E} \) so the right-hand side of the previous solution for \( B \) gets a term \[ \mu_0 \sigma E \frac{\pi a^2}{2(2\pi s)} \] where \( E \) is given above by \( E = \frac{V(t)}{d} \).