**PHYSICS 323: Second midterm May 21, 2019, Solutions**

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1. *Maxwell's Equations and Potentials*

(a) (4) Write down Maxwell's equations in the presence of time-dependent charge and current sources, using $\epsilon_0, \mu_0$ for simplicity.

$$\boldsymbol{\nabla}\cdot\mathbf{E} = \frac{\rho}{\epsilon_0},\ \boldsymbol{\nabla}\cdot\mathbf{B} = 0$$

$$\boldsymbol{\nabla}\times\mathbf{B} = \mu_0\mathbf{J} + \frac{1}{\epsilon_0\mu_0}\frac{\partial\mathbf{E}}{\partial t},\ \boldsymbol{\nabla}\times\mathbf{E} = -\frac{\partial\mathbf{B}}{\partial t}$$

(b) (8) Which of the above Maxwell's equations allows one to derive expressions for $\mathbf{E}(\mathbf{r},t)$ and $\mathbf{B}(\mathbf{r},t)$ in terms of scalar $V(\mathbf{r},t)$ and vector $\mathbf{A}(\mathbf{r},t)$ potentials. Why?

$\boldsymbol{\nabla}\cdot\mathbf{B} = 0$ and $\boldsymbol{\nabla}\times\mathbf{E} = -\frac{\partial\mathbf{B}}{\partial t}$ because these are independent of the sources $\rho$ and $\mathbf{J}$.

(c) (8) Suppose you are given potentials $V(\mathbf{r},t)$ and $\mathbf{A}(\mathbf{r},t)$ that are expressed in the Lorentz gauge. Determine the potentials in the Coulomb gauge.

$\mathbf{E} = -\boldsymbol{\nabla}V - \frac{\partial\mathbf{A}}{\partial t}$, $\mathbf{B} = \boldsymbol{\nabla}\times\mathbf{A}$. Another set of potentials yielding the same fields is given by $\mathbf{A}_C = \mathbf{A} + \boldsymbol{\nabla}\Lambda$, $V_C = V - \frac{\partial\Lambda}{\partial t}$ where $C$ stands for Coulomb. In the Lorentz gauge $\boldsymbol{\nabla}\cdot\mathbf{A} + \frac{1}{c^2}\frac{\partial V}{\partial t} = 0$. In the Coulomb gauge $\boldsymbol{\nabla}\cdot\mathbf{A}_C = 0 = \boldsymbol{\nabla}\cdot(\mathbf{A} + \boldsymbol{\nabla}\Lambda) = -\frac{1}{c^2}\frac{\partial V}{\partial t} + \nabla^2\Lambda$ so $\nabla^2\Lambda = \frac{1}{c^2}\frac{\partial V}{\partial t}$, which means that $\boxed{\Lambda = \frac{-1}{4\pi}c^2\int\frac{\partial V(\mathbf{r}',t)}{\partial t}\,d^3r'}$ Knowing Lambda gives both $V$ and $\mathbf{A}$.

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2. *Vector Potential in the Radiation Zone*

A source of current and charge oscillates with angular frequency $\omega$. The current density is written in complex notation as $\mathbf{J}(\mathbf{r},t) = J_0 e^{-r^2/R^2}\hat{\mathbf{z}}\, e^{-i\omega t}$, where $\hat{\hat{z}}$ represents the direction of the $z$-axis.

(a) (7) Determine the (time-dependent) charge density $\rho(\mathbf{r},t)$.

Use current conservation (CC) $\boldsymbol{\nabla}\cdot\mathbf{J} + \frac{\partial\rho}{\partial t} = 0$. Must have $\rho(\mathbf{r},t) = \rho(\mathbf{r})e^{-i\omega t}$ for CC to hold, so $\frac{\partial\rho}{\partial t} = -i\omega\rho = -\boldsymbol{\nabla}\cdot\mathbf{J} = \frac{\partial J_z(\mathbf{r},t)}{\partial z}$ . Use $\frac{\partial r}{\partial z} = z/r$ to find $\boxed{\rho = \frac{i}{\omega}\frac{z}{r}J_0 e^{-r^2/R^2}\, e^{-i\omega t}}$.

(b) (7) Compute the resulting vector potential $\mathbf{A}(\mathbf{r},t)$ for positions $\mathbf{r}$ such that $r \gg R$. You may express your answer in terms of a well-defined **one**-dimensional integral.

(5) $\mathbf{A}(\mathbf{r},t) = \hat{\mathbf{z}}\frac{\mu_0}{4\pi r}e^{ikr}\int d^3r' e^{i\mathbf{k}\cdot\mathbf{r}'} J_0 e^{-r'^2/R^2}$, with $k = \omega/c$, $\mathbf{k} \equiv k\hat{\mathbf{r}}$

Doing the rest is worth 2.

To get this to a one-dimensional form one needs to evaluate. $\int d\Omega' e^{i\mathbf{k}\cdot\mathbf{r}'}$ let the $z-$ axis be the direction of $\mathbf{k}$. Then $\int d\Omega' e^{i\mathbf{k}\cdot\mathbf{r}'} = \int_0^{2\pi} d\phi \int_0^{\pi} d\theta \sin\theta e^{ik\cos\theta r'} = 4\pi\frac{\sin kr'}{kr'}$ Using this gives

$$\boxed{\mathbf{A}(\mathbf{r},t) = \hat{\mathbf{z}}J_0\frac{\mu_0}{4\pi r}e^{ikr}\int_0^{\infty}\frac{r'dr'}{k}\sin kr' e^{-r'^2/R^2}}$$

(c) (6 ) Determine the electric field for positions along the $z$-axis (for $r \gg R$). Explain your result.

$\boxed{\mathbf{E}=0}$ because $B = \boldsymbol{\nabla}\times\mathbf{A} = i\mathbf{k}\times\mathbf{A}$ and from Maxwell $\mathbf{E} = -c^2/\omega\mathbf{k}\times(\mathbf{k}\times\mathbf{A})$. For positions along the $z$-axis, $\mathbf{k}$ is parallel to the $z$ axis and parallel to $\mathbf{A}$ and $\mathbf{k}\times\mathbf{A} = 0$

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3. *Angular distribution of radiated power.*

Suppose that a source of charge and current, existing in a finite region of space-time, oscillates with an angular frequency $\omega$. Furthermore, measurements made at positions very far from the sources can be summarized by a vector potential given in complex notation by $\mathbf{A}(\mathbf{r},t) = \mu_0 \frac{i\omega}{c} \frac{e^{i(kr-\omega t)}}{r} I_0 L^2 (3\cos^2\theta - 1)\hat{\mathbf{z}}$, where $k = \omega/c$, and $I_0$, $L$ are a given current and length, and $\hat{\mathbf{z}}$, represents the direction of the $z-$axis, and (as usual in spherical coordinates) $\theta$ is the angle between $\mathbf{r}$ and the $z-$axis.

(a) (4) Explain why the factor in the expression for $\mathbf{A}$ is given as $I_0 L^2$.

This is needed for proper dimensions. The vector potential must have units of $\mu_0 I$. The expression here has units of $\mu I_0 L^2 \frac{\omega}{cr}$ $\omega/c$ has units inverse length, so that the given expression has the proper units.

(b) (5) Determine $\mathbf{B}$ for the given vector potential.

$\mathbf{B} = i\mathbf{k} \times \mathbf{A}$, with $\mathbf{k} \equiv k\hat{\mathbf{r}}$ so $\mathbf{B} = (\hat{\mathbf{r}} \times \hat{\mathbf{z}})\mu_0 i (\frac{\omega}{c})^2 \frac{e^{i(kr-\omega t)}}{r} I_0 L^2 (3\cos^2\theta - 1)$ Then finish with $(\hat{\mathbf{r}} \times \hat{\mathbf{z}}) = \hat{\phi}$, so $\boxed{\mathbf{B} = \hat{\phi}\mu_0 i (\frac{\omega}{c})^2 \frac{e^{i(kr-\omega t)}}{r} I_0 L^2 (3\cos^2\theta - 1)}$

(c) (5) Determine $\mathbf{E}$ for the given vector potential. from Maxwell

$\mathbf{E} = ic^2/\omega \mathbf{k} \times \mathbf{B} = ic\hat{\mathbf{r}} \times \mathbf{B} = \boxed{icB\hat{\theta}}$

(d) (6) Determine the angular distribution of radiated power.

$\frac{dP}{d\Omega} = \hat{\mathbf{r}} \cdot \frac{1}{2\mu_0} r^2 (\mathbf{E} \times \mathbf{B}^*) = \frac{c}{2\mu_0}|B|^2 = \frac{c}{2\mu_0}[\mu_0((\frac{\omega}{c})^2 I_0 L^2 (3\cos^2\theta - 1)]^2 (\hat{\mathbf{r}} \times \hat{\mathbf{z}}) \cdot (\hat{\mathbf{r}} \times \hat{\mathbf{z}})$

$(\hat{\mathbf{r}} \times \hat{\mathbf{z}}) \cdot (\hat{\mathbf{r}} \times \hat{\mathbf{z}}) = \sin^2\theta$ so

$$\boxed{\frac{dP}{d\Omega} = \frac{\mu_0 c}{2} (\frac{\omega}{c})^4 I_0^2 L^4 \sin^2\theta (3\cos^2\theta - 1))^2}$$