1. Waves moving through a system with free electrons

Consider an electromagnetic wave of angular frequency \( \omega \) moving through a medium that contains free electrons of charge \( q \) with a density \( n_e \) that is constant in space and time.

(a) (5) An electric field \( \mathbf{E} = E_0(\mathbf{r})e^{-i\omega t} \) of very long wavelength moves through the medium. Use Newton’s equation to determine the current density induced by \( \mathbf{E} \).

Use \( F = m\frac{d\mathbf{v}}{dt} = q\mathbf{E} \) to get \( \frac{d\mathbf{v}}{dt} = \frac{iq}{m}E_0e^{-i\omega t} \). Integrate over \( t \) to get \( \mathbf{v}(t) = \frac{iq}{m}\omega E_0 e^{-i\omega t} \). Then \( \mathbf{j} = qn_e\mathbf{v} = \frac{i}{m}n_eq^2E_0e^{-i\omega t} \).

(b) (5) Write Maxwell’s equations for the electromagnetic wave that moves through the medium.

\[
\nabla \cdot \mathbf{E} = \frac{\rho}{\epsilon_0}, \quad \nabla \cdot \mathbf{B} = 0, \quad \nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}, \quad \nabla \times \mathbf{B} = \mu_0 j + \frac{1}{c^2} \frac{\partial^2 \mathbf{E}}{\partial t^2}.
\]

(c) (5) Use Maxwell’s equations to derive a wave equation for \( \mathbf{E} \).

Take \( \nabla \times \) acting on Faraday’s law so \( \nabla \times (\nabla \times \mathbf{E}) = \nabla(\nabla \cdot \mathbf{E}) - \nabla^2 \mathbf{E} = -\nabla^2 \mathbf{E} \). The last step is obtained from the constant nature of \( \rho \) here. The other side of the equation is \( \frac{\partial}{\partial t} \nabla \times \mathbf{B} = -\frac{\partial}{\partial t} \mu_0 j - \frac{1}{c^2} \frac{\partial^2 \mathbf{E}}{\partial t^2} \). Also \( -\frac{\partial}{\partial t} \mu_0 j = i\omega j = -\frac{\mu_0 q^2}{m} \mathbf{E} \). Then

\[
-\nabla^2 \mathbf{E} + \frac{1}{c^2} \frac{\partial^2 \mathbf{E}}{\partial t^2} + \frac{n_e\mu_0}{m} q^2 \mathbf{E} = 0
\]

(5) Find a condition on the value of \( \omega \) such that the electromagnetic wave can propagate in this medium indefinitely.

Use \( \mathbf{E}_0(\mathbf{r}) = e^{ikz} \) in the equation of the previous problem. Then get \( k^2 - \omega^2/c^2 + \frac{q^2 n_e \mu_0}{m} = 0 \). For propagation \( k \) must take on only real values, so \( k^2 > 0 \). Thus the condition on \( \omega \) is that \( \omega^2 > \frac{c^2 n_e \mu_0}{m} q^2 \).
A waveguide is constructed so that the cross section of the guide form a triangle with sides of length \(a, a\) and \(\sqrt{2}a\) as in the figure. The walls are perfect conductors and \(\epsilon = \epsilon_0, \mu = \mu_0\) inside the guide. Consider only TE electromagnetic waves, and \(B_z\) of the form \(B_z(x, y, z, t) = f(x, y)e^{ikz-\omega t}\).

(a) (5) Determine \(E_z\).

For a transverse electric wave (TE), \(E_z = 0\).

(b) (5) State the explicit boundary conditions that \(f(x, y)\) must obey on each of the three sides of the waveguide.

The normal component of \(B_z\) must vanish on each boundary surface. As shown in class this means that \(\mathbf{n} \cdot \nabla B_z = 0\) on every boundary surface. At \(x = 0\), \(\mathbf{n} = \mathbf{i}\), so \(\partial f(x, y)/\partial x = 0\). At \(y = 0\), \(\mathbf{n} = \mathbf{j}\), so \(\partial f(x, y)/\partial y = 0\). The other boundary is the line with a slope of 1. The line obeys the equation \(x = y\). The unit normal here is \((-\mathbf{i} + \mathbf{j})/\sqrt{2}\), so for the line \(x = y\), \(-\partial f(x, y)/\partial x + \partial f(x, y)/\partial y = 0\).

(c) (5) Determine a set of functions, \(f(x, y)\), that obey the boundary conditions you wrote in part (b). Hint: Try a superposition of the square waveguide solutions.

From the rectangular case shown in class, we should try \(f(x, y) = B_0 \cos \frac{n\pi x}{a} \cos \frac{m\pi y}{a}\), with \(n, m\) integers greater than or equal to 0. This takes care of the boundary conditions at \(x = 0\) and \(y = 0\), but not the ones at \(x = y\).

Following the hint, try \(f(x, y) = B_0 \cos \frac{n\pi x}{a} \cos \frac{m\pi y}{a} + B_0 \cos \frac{m\pi x}{a} \cos \frac{n\pi y}{a}\). First take
\[-\frac{\partial f(x, y)}{\partial x} + \frac{\partial f(x, y)}{\partial y}\] and then after taking the derivatives set \(y = x\). This gives
\[-\frac{\pi B_0}{a} (n \sin \frac{2\pi x}{a} \cos \frac{m\pi x}{a} + m \cos n\pi x \sin \frac{m\pi x}{a} - m \sin \frac{m\pi x}{a} \cos \frac{n\pi x}{a} - n \cos \frac{n\pi x}{a} \sin \frac{2\pi x}{a}) = 0\]

(d) (5) Determine the TE modes for which propagation is allowed for the given value of \(\omega\). The cutoff frequency for a TE mode defined by \(n, m\) is \(\omega_{nm} = \frac{\pi}{ca}\sqrt{m^2 + n^2}\). The necessary values of \(\omega\) obey \(\omega > \omega_{nm}\).
3. Boundary value problem

A plane electromagnetic wave with frequency $\omega$ and wave number $k$ propagates in the $+z$ direction. For $z < 0$ the medium is air with $\epsilon = \epsilon_0$. For $z > 0$ the medium is lossy with $\epsilon = \epsilon_0$ and $\sigma > 0$ for all parts of this problem. For $z > 0$ Maxwell's equations can be manipulated to show that the electric field obeys the equation: $\nabla^2 E - \frac{1}{c^2} \frac{\partial^2 E}{\partial t^2} - \mu_0 \sigma \frac{\partial E}{\partial t} = 0$.

(a) (5) For $z > 0$ determine the relation between $k^2$ and $\omega$.

Use the given partial differential equation with $E = E_0 e^{i(kz - \omega t)}$. This gives $-k^2 + \frac{\omega^2}{c^2} + i \mu_0 \sigma \omega = 0$ so $k = \frac{\omega^2}{c^2} + i \mu_0 \sigma \omega$.

(b) (5) Find the limiting values of $k$ for a very good conductor and a very poor conductor.

For a very good conductor $k^2 \approx i \mu_0 \sigma \omega$, so $k = \frac{(1 + i)}{\sqrt{2}} \sqrt{\mu_0 \sigma \omega}$.

For a very poor conductor $k = \sqrt{\frac{\omega^2}{c^2} + i \mu_0 \sigma \omega} = \frac{\omega}{c} \sqrt{1 + i \frac{\mu_0 \sigma \omega}{2c^2}} \approx \frac{\omega}{c} (1 + i \frac{\mu_0 \sigma \omega}{2c^2})$ So $k \approx \frac{\omega}{c} + i \frac{\sigma}{2 \epsilon_0 c}$.

(c) (5) Find the depth (value of $z$) in the lossy medium such that the electromagnetic wave power is decreased by a factor of $e^{-1}$.

The electric field is damped by a factor $e^{-z \text{Im}[k]} = e^{-z \frac{\sigma}{2 \epsilon_0 c}}$. The power is $\propto$ the absolute square of $E_0$. This means the the relevant depth is $z = \frac{\epsilon_0 c}{\sigma}$.

(d) (5) Find the amplitude of the transmitted electric field to the amplitude of the incident electric field for a very poor conductor.

The transmission coefficient $T$ is given by the ratio of transmitted intensity to incident intensity. For $z < 0$ the wave is a sum of leftward and rightward going waves. For $z > 0$ there is only a rightward-going wave. The incident electric field amplitude is denoted as $E$. The reflected wave has an electric field amplitude $E_R$. The transmitted field has an amplitude $E_T$.

The tangential components of $E$ must be continuous as $z = 0$. So $E_0 + E_R = E_T$ (1). The tangential components of $H = (B/\mu_0$ here) must be continuous. So the boundary condition for $H$ reads $\omega(E_0 - E_R) = (\omega + i \frac{\sigma}{2 \epsilon_0 \omega})E_T$ or $(E_0 - E_R) = (1 + i \frac{\sigma}{2 \epsilon_0 \omega})E_T$. Adding eqns (1) and (2) gives $2E_0 = E_T(2 + \frac{i \sigma}{4 \omega \epsilon_0})$ and $E_T = \frac{E_0}{1 + \frac{i \sigma}{4 \omega \epsilon_0}}$.