PHYSICS 323:
ELECTROMAGNETISM

23 April , 2019 Midterm 1 Solutions

Name: ____ Score __________________/20

1. *Waves moving through a system with free electrons*

Consider an electromagnetic wave of angular frequency ω moving through a medium that contains **free** electrons of charge q with a density n_e that is constant in space and time.

(a) (5) An electric field $\mathbf{E} = \mathbf{E}_0(\mathbf{r})e^{-i\omega t}$ of very long wavelength moves through the medium. Use Newton's equation to determine the current density induced by $\mathbf{E}$.

Use $\mathbf{F} = m\frac{d\mathbf{v}}{dt} = q\mathbf{E}$ to get $\frac{d\mathbf{v}}{dt} = q/m\mathbf{E}_0 e^{-i\omega t}$. Integrate over t to get $\mathbf{v}(t) = \frac{iq\mathbf{E}}{m\omega}e^{-i\omega t}$. Then $\mathbf{j} = qn_e\mathbf{v} = i\frac{n_e q^2 \mathbf{E}_0}{m\omega}e^{-i\omega t}$.

(b) (5) Write Maxwell's equations for the electromagnetic wave that moves through the medium.

$\boldsymbol{\nabla} \cdot \mathbf{E} = \frac{\rho}{\epsilon_0}$, $\boldsymbol{\nabla} \cdot \mathbf{B} = 0$, $\boldsymbol{\nabla} \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$, $\boldsymbol{\nabla} \times \mathbf{B} = \mu_0 \mathbf{j} + \frac{1}{c^2}\frac{\partial \mathbf{E}}{\partial t}$.

(c) (5) Use Maxwell's equations to derive a wave equation for $\mathbf{E}$.

take $\boldsymbol{\nabla}\times$ acting on Faraday's law so $\boldsymbol{\nabla} \times (\boldsymbol{\nabla} \times \mathbf{E}) = \boldsymbol{\nabla}(\boldsymbol{\nabla} \cdot \mathbf{E}) - \nabla^2\mathbf{E} = -\nabla^2\mathbf{E}$. The last step is obtained from the constant nature of ρ here. The other side of the equation is $\frac{\partial}{\partial t}\boldsymbol{\nabla} \times \mathbf{B} = -\frac{\partial}{\partial t}\mu_0\mathbf{j} - \frac{1}{c^2}\frac{\partial^2 \mathbf{E}}{\partial t^2}$. Also $-\frac{\partial}{\partial t}\mu_0\mathbf{j} = i\omega\mathbf{j} = -\frac{\mu_0 n_e q^2}{m}\mathbf{E}$. Then
$-\nabla^2\mathbf{E} + \frac{1}{c^2}\frac{\partial^2\mathbf{E}}{\partial t^2} + \frac{n_e\mu_0}{m}q^2\mathbf{E} = 0$

(5) Find a condition on the value of ω such that the electromagnetic wave can propagate in this medium indefinitely.

Use $\mathbf{E}_0(\mathbf{r}) = e^{ikz}$ in the equation of the previous problem. Then get $k^2 - \omega^2/c^2 + \frac{q^2 n_e \mu_0}{m} = 0$. For propagation k must take on only real values, so $k^2 > 0$. Thus the condition on ω is that $\omega^2 > \frac{c^2 n_e \mu_0}{m}q^2$.

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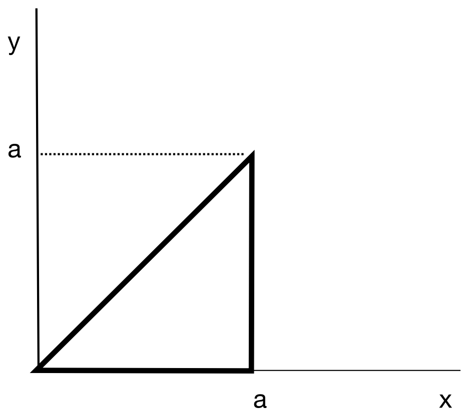
2. *Triangular wave guide*



A waveguide is constructed so that the cross section of the guide form a triangle with sides of length a, a and $\sqrt{2}a$ as in the figure. The walls are perfect conductors and $\epsilon = \epsilon_0$, $\mu = \mu_0$ inside the guide. Consider **only** TE electromagnetic waves, and B_z of the form $B_z(x, y, z, t) = f(x, y)e^{ikz-i\omega t}$.

(a) (5) Determine E_z.

For a transverse electric wave (TE), $E_z = 0$.

(b) (5) State the explicit boundary conditions that $f(x, y)$ must obey on each of the three sides of the waveguide.

The normal component of B_z must vanish on each boundary surface. As shown in class this means that $\mathbf{n} \cdot \boldsymbol{\nabla} B_z = 0$ on every boundary surface. At $x = 0$, $\mathbf{n} = \mathbf{i}$, so $\frac{\partial f(x,y)}{\partial x} = 0$. At $y = 0$, $\mathbf{n} = \mathbf{j}$, so $\frac{\partial f(x,y)}{\partial y} = 0$. The other boundary is the line with a slope of 1. The line obeys the equation $x = y$. The unit normal here is $(-\mathbf{i} + \mathbf{j})/\sqrt{2}$, so for the line $x = y$, $-\frac{\partial f(x,y)}{\partial x} + \frac{\partial f(x,y)}{\partial y} = 0$.

(c) (5) Determine a set of functions, $f(x, y)$, that obey the boundary conditions you wrote in part (b). Hint: Try a superposition of the square waveguide solutions

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From the rectangular case shown in class, we should try $f(x, y) = B_0 \cos\frac{n\pi x}{a} \cos\frac{m\pi y}{a}$, with n, m integers greater than or equal to 0. This takes care of the boundary conditions at $x = 0$ and $y = 0$, but not the ones at $x = y$.

Following the hint, try $\boxed{f(x, y) = B_0 \cos\dfrac{n\pi x}{a} \cos\dfrac{m\pi y}{a} + B_0 \cos\dfrac{m\pi x}{a} \cos\dfrac{ny\pi}{a}}$. First take $-\frac{\partial f(x,y)}{\partial x} + \frac{\partial f(x,y)}{\partial y}$ and then **after** taking the derivatives set $y = x$. This gives $-\frac{\partial f(x,y)}{\partial x} + \frac{\partial f(x,y)}{\partial y} \rightarrow$ $\frac{\pi B_0}{a}\left(n \sin\frac{n\pi x}{a} \cos\frac{m\pi x}{a} + m \cos n\pi x \sin\frac{m\pi x}{a} - m \sin\frac{m\pi x}{a} \cos\frac{n\pi x}{a} - n \cos\frac{m\pi x}{a} \sin\frac{n\pi x}{a}\right) = 0$

(d) (5) Determine the TE modes for which propagation is allowed for the given value of ω.
The cutoff frequency for a TE mode defined by n, m is $\omega_{nm} = \frac{\pi}{ca}\sqrt{m^2 + n^2}$. The necessary values of ω obey $\omega > \omega_{n,m}$.

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3. *Boundary value problem*

A plane electromagnetic wave with frequency ω and wave number k propagates in the $+z$ direction. For $z < 0$ the medium is air with $\epsilon = \epsilon_0$. For $z > 0$ the medium is lossy with $\epsilon = \epsilon_0$ and $\sigma > 0$ for **all** parts of this problem. For $z > 0$ Maxwell's equations can be manipulated to show that the electric field obeys the equation: $\nabla^2 \mathbf{E} - \frac{1}{c^2}\frac{\partial^2 \mathbf{E}}{\partial t^2} - \mu_0 \sigma \frac{\partial \mathbf{E}}{\partial t} = 0$.

(a) (5) For $z > 0$ determine the relation between k^2 and ω.

Use the given partial differential equation with $\mathbf{E} = \mathbf{E}_0 e^{i(kz-\omega t)}$. This gives $-k^2 + \frac{\omega^2}{c^2} + i\mu_0\sigma\omega = 0$ so $\boxed{k^2 = \frac{\omega^2}{c^2} + i\mu_0\sigma\omega}$.

(b) (5) Find the limiting values of k for a very good conductor and a very poor conductor.

For a very good conductor $k^2 \approx i\mu_0\sigma\omega$, so $\boxed{k = \frac{(1+i)}{\sqrt{2}}\sqrt{\mu_0\sigma\omega}}$.

For a very poor conductor $k = \sqrt{\frac{\omega^2}{c^2} + i\mu_0\sigma\omega} = \frac{\omega}{c}\sqrt{1 + i\frac{c^2\mu_0\sigma}{\omega}} \approx \frac{\omega}{c}(1 + i\frac{c^2\mu_0\sigma}{2\omega})$ So $\boxed{k \approx \frac{\omega}{c} + i\frac{\sigma}{2\epsilon_0 c}}$–

(c) (5) Find the depth (value of z) in the lossy medium such that the electromagnetic wave power is decreased by a factor of e^{-1}.

The electric field is damped by a factor $e^{-z\mathrm{Im}[k]} = e^{-z\frac{\sigma}{2\epsilon_0 c}}$ The power is $\propto$ the absolute square of E, or $e^{-z\frac{\sigma}{\epsilon_0 c}}$. This means the the relevant depth is $\boxed{\frac{\epsilon_0 c}{\sigma}}$.

(d) (5) Find the amplitude of the transmitted electric field to the amplitude of the incident electric field for a very poor conductor.

The transmission coefficient T is given by the ratio of transmitted intensity to incident intensity. For $z < 0$ the wave is a sum of leftward and rightward going waves. For $z > 0$ there is only a rightward-going wave. The incident electric field amplitude is denoted as E. The reflected wave has an electric field amplitude E_R. The transmitted field has an amplitude E_T. The tangential components of $\mathbf{E}$ must be continuous as $z = 0$. So $E_0 + E_R = E_T$ (1). The tangential components of $H = (B/\mu_0$ here) must be continuous. So the boundary condition for H reads $\omega(E_0 - E_R) = (\omega + i\frac{\sigma}{2\epsilon_0\omega})E_T$ or $(E_0 - E_R) = (1 + i\frac{\sigma}{2\epsilon_0\omega})E_T(2)$. Adding eqns (1) and (2) gives $2E_0 = E_T(2 + \frac{i\sigma}{2\omega\epsilon_0})$ and

$$\boxed{\frac{E_T}{E_0} = \frac{1}{1 + \frac{i\sigma}{4\omega\epsilon_0}}}$$

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