PHYSICS 322:
ELECTROMAGNETISM

12 June, 2019 Final Exam Solution

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1. *Waves moving through a system with free electrons*

Consider an electromagnetic wave of angular frequency ω moving through a medium that contains **free** electrons of charge q with a *constant density*, n_e, (number of electrons per unit volume) that is constant in space and time.

(a) (6) An electric field $E = E_0(\mathbf{r})e^{-i\omega t}$ of very long wavelength, $\lambda \gg 1/n_e^{1/3}$, moves through the medium. Use Newton's equation to determine the current density induced by $\mathbf{E}$.

Use $\mathbf{F} = m\frac{d\mathbf{v}}{dt} = q\mathbf{E}$ to get $\frac{d\mathbf{v}}{dt} = q/m\mathbf{E}_0 e^{-i\omega t}$. $E_0(\mathbf{r})$ can be considered to be a constant because of the long wavelength approximation. Then integrate over t to get $\mathbf{v}(t) = \frac{iq\mathbf{E}}{m\omega}e^{-i\omega t}$.

Then $\boxed{\mathbf{j} = qn_e\mathbf{v} = i\frac{n_e q^2 \mathbf{E}_0}{m\omega}e^{-i\omega t}}$.

(b) (6) Write Maxwell's equations for the electromagnetic wave that moves through the medium.

$\boldsymbol{\nabla}\cdot\mathbf{E} = \frac{\rho}{\epsilon_0}$, $\boldsymbol{\nabla}\cdot\mathbf{B} = 0$, $\boldsymbol{\nabla}\times\mathbf{E} = -\frac{\partial\mathbf{B}}{\partial t}$, $\boldsymbol{\nabla}\times\mathbf{B} = \mu_0\mathbf{j} + \frac{1}{c^2}\frac{\partial\mathbf{E}}{\partial t}$.

(c) (4) Use the equations of part (b) to derive a wave equation for $\mathbf{E}$. take $\boldsymbol{\nabla}\times$ acting on Faraday's law so $\boldsymbol{\nabla}\times(\boldsymbol{\nabla}\times\mathbf{E}) = \boldsymbol{\nabla}(\boldsymbol{\nabla}\cdot\mathbf{E}) - \nabla^2\mathbf{E} = -\nabla^2\mathbf{E}$. The last step is obtained from the constant nature of ρ here. The other side of the equation is $\frac{\partial}{\partial t}\boldsymbol{\nabla}\times\mathbf{B} = -\frac{\partial}{\partial t}\mu_0\mathbf{j} - \frac{1}{c^2}\frac{\partial^2\mathbf{E}}{\partial t^2}$. Also $-\frac{\partial}{\partial t}\mu_0\mathbf{j} = i\omega\mathbf{j} = -\frac{\mu_0 n_e q^2}{m}\mathbf{E}$. Then $-\nabla^2\mathbf{E} + \frac{1}{c^2}\frac{\partial^2\mathbf{E}}{\partial t^2} + \frac{n_e\mu_0}{m}q^2\mathbf{E} = 0$

(d) (4) Find a condition on the value of ω such that the electromagnetic wave can propagate in this medium indefinitely. Use $\mathbf{E}_0(\mathbf{r}) = e^{ikz}$ in the equation of the previous problem. Then get $k^2 - \omega^2/c^2 + \frac{q^2 n_e \mu_0}{m} = 0$. For propagation k must take on only real values, so $k^2 > 0$. Thus the condition on ω is that $\omega^2 > \frac{c^2 n_e \mu_0}{m}q^2$.

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2. *Relativistic kinematics*

The threshold kinetic energy T_{th} in the laboratory (target particle at rest) for a given reaction is the minimum kinetic energy of the incident particle on a stationary target needed to make the reaction occur.

(a) (6) Calculate the threshold kinetic energy for the process of neutral pion production in a collision between a proton beam and a target proton (which is at rest) so the reaction is: $p + p \to p + p + \pi^0$. Take the proton mass to be M and the pion mass to be m_π.

Four-momentum conservation says $-c^2 P_i^2 = -c^2 P_f^2$ with $P^\mu_{i,f}$ representing the four-momentum and $i(f)$ representing the initial (final) state. $-c^2 P_i^2 = (E_L + Mc^2) - P_L^2 c^2$ with E_L the total Lab energy of the beam $E_L = T_L + Mc^2$ and P_L its momentum and T_L as kinetic energy. $-c^2 P_i^2 = 4Mc^2 + 2T_L Mc^2 = -c^2 P_f^2 = (2M + m_\pi)^2$. The threshold energy is achieved when all particles in the final state are at rest. Solving the previous equation gives

$$\boxed{T_L = m_\pi c^2 + \frac{m_\pi^2}{2M} c^2}$$

(b) (7) An unknown particle of mass M is found to decay into two particles of given mass and momentum $m_1, \mathbf{p}_1$ and $m_2, \mathbf{p}_2$. The magnitude of the momenta of the two final particles and the angle θ between the two particles are measured. Determine the mass M in terms of the given quantities. Four-momentum conservation says $-c^2 P_i^2 = -c^2 P_f^2$. The unknown particle always has $-c^2 P_i^2 = M^2 c^4$ which is equal to $-c^2 P_f^2 = -c^2 (p_1 + p_2)^2$ where $p_{1,2}$ are four-vectors. $-c^2 (p_1 + p_2)^2 = (E_1 + E_2)^2 - c^2 (\mathbf{p}_1 + \mathbf{p}_2)^2 = E_1^2 - c^2 \mathbf{p}_1^2 + E_2^2 - c^2 \mathbf{p}_2^2 + 2E_1 E_2 - 2c^2 |\mathbf{p}_1| \, |\mathbf{p}_2| \cos\theta$ so $\boxed{M^2 = M_1^2 + M_2^2 + 2E_1 E_2 / c^4 - 2|\mathbf{p}_1| \, |\mathbf{p}_2| \cos\theta / c^2}$, where $E_{1,2} = \sqrt{M_{1,2}^2 c^4 + \mathbf{p}_{1,2}^2 c^2}$

(c) (7) For the situation of (b) the particle of mass M is created in a lab with a kinetic energy of Mc^2. Suppose it has a known lifetime of τ (in its rest frame). How far on average will it travel in the lab before decaying?

The travel time in the lab is $D = v\gamma(v)\tau$.The total energy is $2Mc^2 = Mc^2\gamma(v)$, so $\gamma = 2$. The speed of the particle is given by $\gamma = 1/\sqrt{1 - v^2/c^2}$, so $v/c = \sqrt{3}/2$. The distance travelled is $= \sqrt{3}/2c * 2 * \tau = \sqrt{3} c\tau$

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3. *Four-Vector Potential*

In a frame $\mathcal{S}$, the vector potential $\mathbf{A}$ is given by $\mathbf{A} = A_0\hat{\mathbf{z}}\exp(-r^2/R^2)$, where R is a given length and A_0 is a given potential strength, a constant with dimensions of Tm. The electric potential $V(\vec{r}, t) = 0$ in that frame.

(a) (7) Determine the charge density ρ in the frame $\mathcal{S}$.

$(\frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \boldsymbol{\nabla}^2)V \propto \rho = 0$, so with the scalar potential $V = 0$ $\boxed{\rho(\mathbf{r}, t) = 0}$

(b) (7) Determine $\mathbf{E}(\mathbf{r}, \mathbf{t})$ and $\mathbf{B}(\mathbf{r}, \mathbf{t})$ in $\mathcal{S}$. $\boxed{\mathbf{E} = -\dfrac{\partial \mathbf{A}}{\partial t} = 0}$, $\mathbf{B} = \boldsymbol{\nabla} \times \mathbf{A}$,

$B_x = A_0\frac{\partial}{\partial y}e^{-r^2/R^2} = -2A_0\frac{y}{R^2}e^{-r^2/R^2}$, $B_y = -A_0\frac{\partial}{\partial x}e^{-r^2/R^2} = -2A_0\frac{x}{R^2}e^{-r^2/R^2}$,

(c) (6) The frame $\overline{\mathcal{S}}$ moves relative to $\mathcal{S}$ at a velocity $\mathbf{v} = v\hat{x}$, and its origin overlaps that of $\mathcal{S}$ at $t = 0$. Determine the four-vector potential in the frame $\bar{S}$: $\bar{A}^\mu(\bar{\mathbf{r}}, \bar{t})$.

$x = \gamma(\bar{x} + v\bar{t})$, $y = \bar{y}$, $z = \bar{z}$, $t = \gamma(\bar{t} + v/c^2\bar{x})$, $\bar{r}^2 = \gamma^2(\bar{x} + v\bar{t})^2 + \bar{y}^2 + \bar{z}^2$, $\gamma = 1/\sqrt{1 - v^2/c^2}$

$\boxed{\bar{A}^0(\bar{\mathbf{r}}, \bar{t}) = \gamma(A^0 + v/cA^1) = 0, \bar{A}^1(\bar{\mathbf{r}}, \bar{t}) = \gamma(A^1 + vA^0) = 0,\ \bar{A}^3(\bar{\mathbf{r}}, t) = A^3 = A_0e^{-\bar{r}^2/R^2}}$

(d)(5) Determine the electric field $\overline{\mathbf{E}}(\bar{\mathbf{r}}, \bar{t})$, in the frame $\overline{\mathcal{S}}$.

$\bar{E} = -\frac{\partial \bar{A}}{\partial \bar{t}} = \gamma v\frac{2A_0}{R^2}(\bar{x} + v\bar{t})e^{-\bar{r}^2/R^2}$

Can also use field transformations of Chap 12

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4. *Circular antenna*

An antenna consists of a circular loop of wire of radius R located in the $x-y$ plane with its center at the origin. The current in the wire is $I = I_0 \cos\omega t$. Consider positions in the radiation zone.

(a) (5) Show that (using complex notation) $\mathbf{A}(\mathbf{r},t) = \frac{\mu_0}{4\pi}\frac{I_0}{r}\mathbf{K}(k,\theta)e^{-i\omega(t-r/c)}$, define the quantity k and obtain an expression for $\mathbf{K}(k,\theta)$ as a well-defined one-dimensional integral. Hint: you may take the vector $\mathbf{r}$ to lie in the xz plane (with $y=0$).
$\mathbf{A}(\mathbf{r},t) = \frac{\mu_0}{4\pi}\oint d\mathbf{l}' I(\mathbf{r}', t - |\mathbf{r}-\mathbf{r}'|/c)$ In the radiation zone $|\mathbf{r}-\mathbf{r}'| = r - \hat{\mathbf{r}}\cdot\mathbf{r}'$ so then with complex notaton

$\mathbf{A}(\mathbf{r},t) = \frac{\mu_0}{4\pi r}I_0\oint d\mathbf{l}' e^{-i\omega(t-r/c+\hat{\mathbf{r}}\cdot\mathbf{r}'} = \frac{\mu_0}{4\pi r}I_0 e^{-i\omega(t-r/c)}\oint d\mathbf{l}' e^{-i\mathbf{k}\cdot\mathbf{r}'}$, where $\mathbf{k} = k\hat{\mathbf{r}}$, with $\boxed{k = \omega/c}$.

Put the $x-$ axis under $\mathbf{r}$ so that $\mathbf{r} = \sin\theta\hat{\mathbf{i}} + \cos\theta\hat{\mathbf{z}}$. Then
$d\mathbf{l}' = Rd\phi'\hat{\phi}' = Rd\phi'(-\sin\phi'\hat{\mathbf{x}} + \cos\phi'\hat{\mathbf{y}},\ \mathbf{k}\cdot\mathbf{r}' = \omega/cR\sin\theta\cos\phi')$. Thus
$\mathbf{A}(\mathbf{r},t) = \frac{\mu_0}{4\pi}RI_0 e^{-i(\omega t - r/c)}\int_0^{2\pi} d\phi'(-\sin\phi'\hat{\mathbf{x}} + \cos\phi'\hat{\mathbf{y}})e^{-i\omega R/c\sin\theta\cos\phi'}$

Then $\boxed{\mathbf{K} = R\int_0^{2\pi} d\phi'(-\sin\phi'\hat{\mathbf{x}} + \cos\phi'\hat{\mathbf{y}})e^{-ikR\sin\theta\cos\phi'}}$

(b) (10) Express $\mathbf{E}$, $\mathbf{B}$ and $\frac{dP}{d\Omega}$ in terms of $\mathbf{K}(k,\theta)$.
$\mathbf{B} = \boldsymbol{\nabla}\times\mathbf{A} = ik\hat{\mathbf{r}}\times\mathbf{A} = \frac{ik\mu_0}{4\pi r}I_0 e^{-i(\omega t - r/c)}\hat{\mathbf{r}}\times\mathbf{K}$

$\mathbf{E} = c\mathbf{B}\times\hat{\mathbf{r}} = \frac{ik\mu_0}{4\pi r}I_0 e^{-i(\omega t - r/c)}(\mathbf{I} - \hat{\mathbf{r}}\,\hat{\mathbf{r}}\cdot\mathbf{K})$

$\frac{dP}{d\Omega} = \frac{1}{2}Re(\mathbf{E}\times\mathbf{B}^*\mu_0)\cdot\hat{\mathbf{r}}r^2 = \boxed{\frac{ck^2}{2\mu_0}(\frac{\mu_0 I_0}{4\pi})^2(|\mathbf{K}|^{\mathbf{2}} - (\hat{\mathbf{r}}\cdot\mathbf{K})^{\mathbf{2}})}$

(c) (5) In general kR can take on any value, so it is necessary to evaluate $\mathbf{K}(k,\theta)$ **without approximation**. This can be done using the relation:
$e^{iks\cos\phi} = J_0(ks) + 2\sum_{m=1}^{m=\infty} i^m e^{im\phi}J_m(ks)$, where $J_m(x)$ is a cylindrical Bessel function.
Obtain $\mathbf{K}(k,\theta)$ without approximation in terms of a J_m, specifying the argument. There are two angular integrals in the expression for $\mathbf{I}$. The $\hat{\mathbf{x}}$ component vanishes because of cancellation. The integral from 0 to π is cancelled by the one from π to 2π.The remaining integral is $\mathbf{K} = \hat{\mathbf{y}}R\int_0^{2\pi} d\phi'\cos\phi'\,(J_0(kR) + 2\sum_{m=1}^{m=\infty} i^m e^{im\phi'}J_m(kR))$. The term with $m=1$ is the only one that is non-zero. $\int_0^{2\pi} d\phi'\cos\phi'\, i^m e^{im\phi'} = i2\pi 1/2$ so $\boxed{\mathbf{K} = \hat{\mathbf{y}}Ri\pi J_1(kR)}$