1. Waves moving through a system with free electrons

Consider an electromagnetic wave of angular frequency $\omega$ moving through a medium that contains free electrons of charge $q$ with a constant density, $n_e$, (number of electrons per unit volume) that is constant in space and time.

(a) (6) An electric field $E = E_0(r)e^{-i\omega t}$ of very long wavelength, $\lambda \gg 1/n_e^{1/3}$, moves through the medium. Use Newton’s equation to determine the current density induced by $E$.

Use $\mathbf{F} = m \frac{d\mathbf{v}}{dt} = q\mathbf{E}$ to get $\frac{d\mathbf{v}}{dt} = \frac{q}{m}E_0 e^{-i\omega t}$. $E_0(r)$ can be considered to be a constant because of the long wavelength approximation. Then integrate over $t$ to get $\mathbf{v}(t) = \frac{iq}{m\omega}E_0 e^{-i\omega t}$.

Then $\mathbf{j} = qn_e \mathbf{v} = i\frac{n_e q^2 E_0}{m\omega} e^{-i\omega t}$.

(b) (6) Write Maxwell’s equations for the electromagnetic wave that moves through the medium.

$$\nabla \cdot \mathbf{E} = \frac{\rho}{\epsilon_0}, \quad \nabla \cdot \mathbf{B} = 0, \quad \nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}, \quad \nabla \times \mathbf{B} = \mu_0 \mathbf{j} + \frac{1}{ct^2} \frac{\partial \mathbf{E}}{\partial t}.$$ 

(c) (4) Use the equations of part (b) to derive a wave equation for $\mathbf{E}$, take $\nabla \times$ acting on Faraday’s law so $\nabla \times (\nabla \times \mathbf{E}) = \nabla(\nabla \cdot \mathbf{E}) - \nabla^2 \mathbf{E} = -\nabla^2 \mathbf{E}$. The last step is obtained from the constant nature of $\rho$ here. The other side of the equation is $\frac{\partial}{\partial t} \nabla \times \mathbf{B} = -\frac{\partial}{\partial t} \mu_0 \mathbf{j} - \frac{1}{ct^2} \frac{\partial^2 \mathbf{E}}{\partial t^2}$. Also $-\frac{\partial}{\partial t} \mu_0 \mathbf{j} = i\omega \mathbf{j} = -\frac{\mu_0 n_e q^2}{m} \mathbf{E}$. Then $-\nabla^2 \mathbf{E} + \frac{1}{c^2} \frac{\partial^2 \mathbf{E}}{\partial t^2} + \frac{n_e \mu_0}{m} q^2 \mathbf{E} = 0$.

(d) (4) Find a condition on the value of $\omega$ such that the electromagnetic wave can propagate in this medium indefinitely. Use $\mathbf{E}_0(\mathbf{r}) = e^{ikz}$ in the equation of the previous problem. Then get $k^2 - \omega^2/c^2 + \frac{q^2 n_e \mu_0}{m} = 0$. For propagation $k$ must take on only real values, so $k^2 > 0$. Thus the condition on $\omega$ is that $\omega^2 > \frac{c^2 n_e \mu_0}{m} q^2$. 

1
2. Relativistic kinematics

The threshold kinetic energy $T_{th}$ in the laboratory (target particle at rest) for a given reaction is the minimum kinetic energy of the incident particle on a stationary target needed to make the reaction occur.

(a) (6) Calculate the threshold kinetic energy for the process of neutral pion production in a collision between a proton beam and a target proton (which is at rest) so the reaction is: $p + p \rightarrow p + p + \pi^0$. Take the proton mass to be $M$ and the pion mass to be $m_\pi$.

Four-momentum conservation says $-c^2 P_{i}^2 = -c^2 P_{f}^2$ with $P_{i,f}^\mu$ representing the four-momentum and $i(f)$ representing the initial (final) state. $-c^2 P_{i}^2 = (E_L + M c^2) - P_L^2 c^2$ with $E_L$ the total Lab energy of the beam $E_L = T_L + M c^2$ and $P_L$ its momentum and $T_L$ as kinetic energy. $-c^2 P_{f}^2 = 4 M c^2 + 2 T_L M c^2 = -c^2 P_{f}^2 = (2M + m_\pi)^2$. The threshold energy is achieved when all particles in the final state are at rest. Solving the previous equation gives

$$T_L = m_\pi c^2 + \frac{m_\pi^2}{2M}$$

(b) (7) An unknown particle of mass $M$ is found to decay into two particles of given mass and momentum $m_1, \mathbf{p}_1$ and $m_2, \mathbf{p}_2$. The magnitude of the momenta of the two final particles and the angle $\theta$ between the two particles are measured. Determine the mass $M$ in terms of the given quantities. Four-momentum conservation says $-c^2 P_{i}^2 = -c^2 P_{f}^2$. The unknown particle always has $-c^2 P_{i}^2 = M^2 c^4$ which is equal to $-c^2 P_{f}^2 = -c^2 (p_1 + p_2)^2$ where $p_{1,2}$ are four-vectors. $-c^2 (p_1 + p_2)^2 = (E_1 + E_2)^2 - c^2 (p_1 + p_2)^2 = E_1^2 - c^2 p_1^2 + E_2^2 - c^2 p_2^2 + 2E_1 E_2 - 2c^2 |\mathbf{p}_1||\mathbf{p}_2| \cos \theta$ so

$$M^2 = M_1^2 + M_2^2 + 2E_1 E_2/c^4 - 2|\mathbf{p}_1||\mathbf{p}_2| \cos \theta/c^2$$

where $E_{1,2} = \sqrt{M_{1,2}^2 c^4 + \mathbf{p}_{1,2}^2 c^2}$

(c) (7) For the situation of (b) the particle of mass $M$ is created in a lab with a kinetic energy of $M c^2$. Suppose it has a known lifetime of $\tau$ (in its rest frame). How far on average will it travel in the lab before decaying?

The travel time in the lab is $D = v \gamma(v) \tau$. The total energy is $2M c^2 = M c^2 \gamma(v)$, so $\gamma = 2$. The speed of the particle is given by $\gamma = 1/\sqrt{1 - v^2/c^2}$, so $v/c = \sqrt{3}/2$. The distance travelled is

$$D = \frac{\sqrt{3}}{2c} \cdot 2 \cdot \tau = \sqrt{3}c \tau$$
3. **Four-Vector Potential**

In a frame $\mathcal{S}$, the vector potential $\mathbf{A}$ is given by $\mathbf{A} = A_0 \hat{z} \exp(-r^2/R^2)$, where $R$ is a given length and $A_0$ is a given potential strength, a constant with dimensions of $\text{Tm}$. The electric potential $V(\vec{r}, t) = 0$ in that frame.

(a) (7) Determine the charge density $\rho$ in the frame $\mathcal{S}$.

$$(\frac{1}{c^2} \frac{\partial^2}{\partial t^2} - \nabla^2) V \propto \rho = 0,$$

so with the scalar potential $V = 0$ we have $\rho(r, t) = 0$.

(b) (7) Determine $\mathbf{E}(\mathbf{r}, t)$ and $\mathbf{B}(\mathbf{r}, t)$ in $\mathcal{S}$.

$$\mathbf{E} = -\frac{\partial \mathbf{A}}{\partial t} = 0, \quad \mathbf{B} = \nabla \times \mathbf{A},$$

$$E_x = A_0 \frac{\partial}{\partial y} e^{-r^2/R^2} = -2A_0 \frac{y}{R^2} e^{-r^2/R^2}, \quad B_y = -A_0 \frac{\partial}{\partial x} e^{-r^2/R^2} = -2A_0 \frac{x}{R^2} e^{-r^2/R^2},$$

(c) (6) The frame $\mathcal{S}$ moves relative to $\mathcal{S}$ at a velocity $\mathbf{v} = v\hat{x}$, and its origin overlaps that of $\mathcal{S}$ at $t = 0$. Determine the four-vector potential in the frame $\mathcal{S}'$: $\vec{A}'(\vec{r}, \bar{t})$.

$$x = \gamma(x + vt), \quad y = \bar{y}, \quad z = \bar{z}, \quad \bar{t} = \gamma(t + v/c^2 \bar{x}), \quad \bar{r}^2 = \gamma^2 (x + vt)^2 + \bar{y}^2 + \bar{z}^2, \quad \gamma = 1/\sqrt{1 - v^2/c^2}$$

$$\vec{A}'(\vec{r}, \bar{t}) = \gamma(A^0 + v/c A^1) = 0, \quad \bar{A}^1(\vec{r}, \bar{t}) = \gamma(A^1 + v A^0) = 0, \quad \bar{A}^3(\vec{r}, t) = A^3 = A_0 e^{-r^2/R^2}$$

(d)(5) Determine the electric field $\vec{E}(\vec{r}, \bar{t})$, in the frame $\mathcal{S}$.

$$\vec{E} = -\frac{\partial \mathbf{A}}{\partial \bar{t}} = \gamma v \frac{\partial A_0}{\partial \bar{r}} (x + vt) e^{-r^2/R^2}$$

Can also use field transformations of Chap 12.
4. Circular antenna

An antenna consists of a circular loop of wire of radius $R$ located in the $x-y$ plane with its center at the origin. The current in the wire is $I = I_0 \cos \omega t$. Consider positions in the radiation zone.

(a) (5) Show that (using complex notation) $A(r, t) = \frac{i \mu_0 I_0}{4\pi} K(k, \theta)e^{-i\omega(t-r/c)}$, define the quantity $k$ and obtain an expression for $K(k, \theta)$ as a well-defined one-dimensional integral. Hint: you may take the vector $\mathbf{r}$ to lie in the $xz$ plane (with $y = 0$).

\[ A(r, t) = \frac{i \mu_0 I_0}{4\pi} \oint d\mathbf{l} e^{-i\omega(t-r/c)} = \frac{i \mu_0 I_0}{4\pi} \oint d\mathbf{l} e^{-i\mathbf{k} \cdot \mathbf{r}'} \]

Put the $x-$axis under $\mathbf{r}$ so that $\mathbf{r} = \sin \theta \hat{x} + \cos \theta \hat{z}$. Then

\[ d\mathbf{l}' = Rd\phi \hat{\phi}' = Rd\phi'(-\sin \phi' \hat{x} + \cos \phi' \hat{y}) \]

Thus

\[ A(r, t) = \frac{i \mu_0 I_0}{4\pi} R\Omega e^{-i(\omega - \pi/2)\epsilon_0 k_0} J_0(kR) \]

Then

\[ K = R \int_0^{2\pi} d\phi' (-\sin \phi' \hat{x} + \cos \phi' \hat{y}) e^{-i\pi R \sin \theta \cos \phi'} \]

(b) (10) Express $\mathbf{E}$, $\mathbf{B}$ and $\frac{d\mathbf{P}}{dt}$ in terms of $K(k, \theta)$.

\[ \mathbf{B} = \nabla \times \mathbf{A} = ik\hat{x} \times \mathbf{A} = \frac{i \mu_0 I_0}{4\pi} e^{-i(\omega - \pi/2)kR} (\hat{x} \times \hat{K}) \]

\[ \mathbf{E} = \mathbf{cB} \times \hat{r} = \frac{i \mu_0 I_0}{4\pi} e^{-i(\omega - \pi/2)kR} (\mathbf{I} - \hat{r} \cdot \hat{K}) \]

\[ \frac{d\mathbf{P}}{dt} = \frac{1}{2} \Re(\mathbf{E} \times \mathbf{B}^* \mu_0) \cdot \hat{r} r^2 = \frac{ck^2}{2\mu_0} \left( \frac{\mu_0 I_0}{4\pi} \right)^2 (|\mathbf{K}|^2 - (\hat{r} \cdot \hat{K})^2) \]

(c) (5) In general $kR$ can take on any value, so it is necessary to evaluate $K(k, \theta)$ without approximation. This can be done using the relation:

\[ e^{ikx \cos \phi} = J_0(kx) + \sum_{m=1}^{\infty} i^m e^{im\phi} J_m(kx) \]

Obtain $K(k, \theta)$ without approximation in terms of a $J_m$, specifying the argument. There are two angular integrals in the expression for I. The $\hat{x}$ component vanishes because of cancellation. The integral from 0 to $\pi$ is cancelled by the one from $\pi$ to $2\pi$. The remaining integral is $K = \hat{y} R \int_0^{2\pi} d\phi' \cos \phi' (J_0(kR) + \sum_{m=1}^{\infty} i^m e^{im\phi} J_m(kR))$. The term with $m = 1$ is the only one that is non-zero. $\int_0^{2\pi} d\phi' \cos \phi' i^m e^{im\phi} = i2\pi 1/2$ so

\[ K = \hat{y} R i\pi J_1(kR) \]