Micro waves are important in current elec. eng. 300 MHz ≤ ν ≤ 360 GHz 1 mm ≤ λ ≤ 1 m

ability to generate microwaves & control intensity → new technologies used everyday.

First use - radar

* WWII military: nav
* Air Traffic Control
* Obs Weather
* Speed limits

Basic idea: detect object by scattering of waves - λ ~ size of object

More uses

* Cell phone system (1981) ν 824-849 MHz & 879-894 MHz

* Microwave links between antennas: court TV & telephone (470-890 MHz)

* Microwave oven 2.45 GHz

* Cosmic scale - radiation remnant of big bang is black body spectrum at T = 3K, max in microwave range
Wave guides are an essential component of microwave circuits - hollow tubes in which EM waves propagate with little attenuation. Radar waves generated in a resonant cavity travel through a guide to an antenna from which emerge in free space to be scattered by detected objects.

Wave

Conducting walls

Vacuum or dielectric

Want to study how waves propagate.

Complication - fields make electrons in walls (boundaries) oscillate and extra waves are generated.

Radiation sent into a guide reflects from any conducting wall that it hits so that it bounces back and forth.
Fields in guides between walls. Fields in guides between walls influence each other. Math - PDE ω

Boundary condition

The math is similar to quantum mechanics - only discrete modes can propagate

E & M 6 quantities E, B, ψ

The walls will be assumed to be perfect conductors \( \sigma \to \infty \) conductivity

skin depth is 0 E & B don't penetrate metal. Only on surface

Logic

\( \mathbf{E} = 0 \) inside perfect conductor

Faraday \( \mathbf{E} / \partial t = 0 \) assume \( \mathbf{B} = 0 \) initially

Stays 0

Goal - compute \( \mathbf{E} \), \( \mathbf{B} \) that propagate boundary value problem

\[ \text{Boundary} \]

\[ \text{Value} \]

\[ \text{Problem} \]

\[ \text{interior} \]

\[ \text{Wall} \]
Bound any conditions at surface

Tangential \( \vec{E} \) is continuous

\[ \hat{n} \times (\vec{E}_1 - \vec{E}_2) = 0 \quad E_2 = 0 \]

\[ \hat{n} \times \vec{E}_1 = 0 \] Components of \( \vec{E} \) \( \parallel \) to surface normal component \( \vec{B} \) is continuous \( \nabla \cdot \vec{B} = 0 \)

\[ \hat{n} \cdot (\vec{B}_1 - \vec{B}_2) = 0 \Rightarrow \hat{n} \cdot \vec{B}_1 = 0 \]

Comp of \( \vec{B} \) \( \perp \) to surface = 0

So to summarize \( E_\parallel = 0 \) \( B_\perp = 0 \) at surface.

Have hollow conductor - propagation in 2 direction. Assume harmonic time dependence

\( \vec{E}(x,y,z,t) = \vec{E}(x,y,z) \ e^{-i\omega t} \)

\( \vec{B}(x,y,z,t) = \vec{B}(x) \ e^{-i\omega t} \)

Maxwell \( \hat{n} \cdot \vec{E} = 0 \) \( \hat{n} \cdot \vec{B} = 0 \)

\[ \nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} \quad \nabla \times \vec{B} = \frac{1}{c^2} \frac{\partial^2 \vec{E}}{\partial t^2} = -\frac{\omega^2}{c^2} \vec{E} \]

If medium is not hollow \( \frac{1}{c^2} \rightarrow \frac{\mu \epsilon}{\mu_0 \epsilon_0} \)

\[ M_0 = \mu_0 \]

Take \( \nabla \times (\nabla \times \vec{E}) \) as \( \nabla \times \vec{B} \rightarrow \) wave eq
\[
\left( \nabla^2 + \frac{\omega^2}{c^2} \right) \left\{ \frac{E}{B} \right\} = 0
\]

Assume:

\[
E = E_z + E_t
\]

\[
E_z = \frac{1}{2} \cdot E
\]

Same notation for \( B \)

Waveguide shape is same for all values of \( z \)

\( t = \text{transverse to prop direction} \)

Want waves to propagate in \( z \)-direction

\[
E(x, y, z) = E(x, y) e^{i k z}
\]

\[
B(x, y, z) = B(x, y) e^{i k z}
\]

\[
\nabla^2 \Rightarrow \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} - k^2 = \nabla^2 - k^2
\]

So:

\[
\left( \nabla^2 + \frac{\omega^2}{c^2} - k^2 \right) \left\{ \frac{E(x, y)}{B(x, y)} \right\} = 0
\]

What kind of waves can propagate:

1) TE Transverse Electric \( E_2 = 0 \)

2) TM " Magnetic \( B_2 = 0 \)

General wave sums of both kinds

3) TEM \( E_2 = 0 \) \( B_2 = 0 \)

This happens in free space now situation is different because of BC
\[ E_2 = 0, \quad B_2 = 0 \]

We have \( \nabla \cdot \vec{E} = 0 \) with \( B_z = 0 \) and \( E_z = 0 \). Take 2-component of \( \nabla \times \vec{E} = \frac{\mu_0}{c} \vec{B} \).

So 2-comp. of \( \nabla \times \vec{E} = 0 \) is 0.

So \( \nabla \times \vec{E} = 0 \)

with \( E_2 = 0 \) \( \nabla \times \vec{E} = 0 \)

Both \( \nabla \cdot \vec{E} = 0 \) \( \nabla \times \vec{E} = 0 \) so

\( \vec{E} \) must vanish no field at all.

Note: if there is a separate conductor in the middle

there will be TEM mode. For now have TE 0 TM solutions.

Goal: compute allowed modes.

That so we BC \( \emptyset \) \( \text{WE} \).

**Summary**

\[ \vec{E} (x, y) e^{1/bz - iwt} \]

TE \( E_2 = 0 \) TM \( B_2 = 0 \)

Surface Boundary Conditions:

1) \( E_{y} = 0 \) \( E_2 = 0 \) 2) \( \nabla \cdot \vec{B} = 0 \)

\( \nabla^2 + \omega^2 - k_0^2 \) \( \vec{E} \) = 0
Solution strategy

**TE** or **TM**

Solve wave eq. for \( B_2 \)

Use Maxwell to get \( E_x, E_y, B_x, B_y \) from \( B_2 \)

**TM;** solve wave eq. for \( E_2 \)

Use Maxwell to get \( E_x, E_y, B_x, B_y \) from \( E_2 \)

Do **TM** first

\[
\left( \nabla^2 + \frac{\omega^2}{c^2} - k_0^2 \right) E_2 = 0 \quad \Rightarrow \quad \left( \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\omega^2}{c^2} - k_0^2 \right) E_2 = 0 \quad (1)
\]

with \( E_2 = 0 \) on surface

To be specific rectangular wave guide

\[
E_2 = X(x) Y(y)
\]

with \( X(x=0,a)=0, \ Y(y=0,a)=0 \)

\[
\frac{\partial^2 Y}{\partial x^2} \frac{dX}{dx} + X \frac{d^2 Y}{dy^2} = \frac{(\omega^2 + k_0^2)XY}{c^2}
\]

Divide by \( XY \)

\[
\frac{1}{X} X'' + \frac{1}{Y} Y'' = \frac{\lambda_0^2 - \omega^2}{c^2} \text{ const}
\]

Each of \( \frac{X''}{X}, \frac{Y''}{Y} \) must be constant
The constant must be negative be cause the BC repeat

\[ X = \sin k_1 x, \quad X'' = -k_1^2 \]

with \[ k_1 a = n\pi \]
not all \[ k_1 \] work

\[ Y = \sin k_2 y, \quad Y'' = -k_2^2 \]

\[ k_2 b = m\pi \]

And \[ -k_1^2 - k_2^2 = \frac{\omega^2}{c^2} \]

from (1)

\[ k_1^2 = \frac{\omega^2}{c^2} - (\frac{n\pi}{a})^2 - (\frac{m\pi}{b})^2 \]

\[ k_1^2 \text{ must be } > 0 \text{, from the real factor for prop} \]

A mode is defined by the numbers \( n, m \)

For a given \( n, m \)

\[ \omega > c \sqrt{\left(\frac{n\pi}{a}\right)^2 + \left(\frac{m\pi}{b}\right)^2} = \omega_{n,m} \]

\( \omega_{n,m} \) is cutoff frequency of mode

Now \( \therefore \) Thus \( E_z = E_0 \sin n\pi x \sin m\pi y e^{-i\omega t} \) get other components
How? Maxwell eq

$$\nabla \times \vec{B} = -\frac{\omega}{c^2} \vec{E}$$

$$\nabla \times \vec{E} = \omega \vec{B}$$

We need to separate transverse and \( \frac{\partial}{\partial t} \) components.

Take \( \frac{\partial}{\partial t} \times \) these eqns

Use vector identity: any vector \( \vec{V} \)

$$\nabla \times (\nabla \times \vec{V}) = \nabla (\nabla \cdot \vec{V}) - \frac{\partial}{\partial t} \nabla \vec{V}$$

For \( \vec{B}_t \)

$$\nabla \times (\nabla \times \vec{B}_t) = \nabla \nabla \cdot \vec{B}_t - \frac{\partial}{\partial t} \vec{B}_t = -\frac{\omega}{c^2} \nabla \times \vec{B}_t$$

$$\vec{B}_t = \frac{\omega}{c^2} \nabla \times \vec{E}_t$$

For \( \vec{E}_t \)

$$\nabla \times (\nabla \times \vec{E}_t) = \nabla (\nabla \cdot \vec{E}_t) - \frac{\partial}{\partial t} \nabla \vec{E}_t$$

\[ \nabla \cdot \vec{E}_t = \frac{i \hbar}{\omega^2 - \omega^2} (\nabla \times 0) - \frac{i \hbar}{\omega^2 - \omega^2} (-\vec{E}_t) \]

A vector equation. Z component is \( 0 = 0 \) physics is not.

Transverse component

$$\vec{E}_t = \frac{i \hbar}{\omega^2 - \omega^2} \nabla_{t \perp} B_2 (x, y) = \frac{i \hbar}{\omega^2 - \omega^2} \nabla_{\perp} E_2$$
Last time we discussed waveguides.

**Summary**

- Conductivity \( \rightarrow \infty \)

Waves propagate as \( e^{-i(\omega t - kz)} \)

Mode: \( TE_1 \) yesterday, \( B_z = 0 \)

\[ \text{Ideas: Maxwell's eq} \rightarrow \text{wave eq} \]

\[ \left( \nabla^2 + \omega^2 - \frac{n^2}{c^2} \right) \left\{ \frac{\hat{E}}{\hat{B}} \right\} = 0 \]

\[ \nabla^2 \hat{E} = \frac{\partial^2 \hat{E}}{\partial x^2} + \frac{\partial^2 \hat{E}}{\partial y^2} + \frac{\partial^2 \hat{E}}{\partial z^2} \]

**BC**

- \( E_{11} \) to surface = 0
- \( B_z \) to surface = 0

Found some called modes.

Get \( E_x, B_y, B_z \) from \( E_z \)

Today: continue with more interesting case.

**TE**

- \( E_z = 0 \)
- \( \omega_{TE}^2 = \omega^2 - \omega_{ni}^2 \)

\[ \omega_{ni} = \sqrt{ \left( \frac{\omega_{ni}^2}{a} \right)^2 + \left( \frac{\omega_{ni}^2}{b} \right)^2} \]

**Cut-off**


**TE modes**

First get $B_z$

This is more difficult than TM because $\mathbf{A} \cdot \mathbf{B} = 0$.

$B_z = 0$ on surface must get $B_y$ in terms of $B_z$.

There are two main BC's in UG physics function $= 0$ or function $' = 0$.

It is natural to expect that $\mathbf{\hat{n}} \cdot \nabla_x B_z = 0$ on surface.

This is because all solutions are linear combos of TE & TM and TM uses $E_z |_s = 0$.

$\mathbf{\hat{n}} \cdot \nabla_x$ is either $\frac{\partial}{\partial x}$ or $\frac{\partial}{\partial y}$.

The actual BC is that $\mathbf{\hat{n}} \cdot \mathbf{B} = 0 = \mathbf{\hat{n}} \cdot \mathbf{B}_y |_s$.

Need to get condition on $B_z$ from this.
\[ \mathbf{\nabla} \times \mathbf{E} = \omega \mathbf{B} \quad \mathbf{\nabla} \times \mathbf{B} = -\frac{\partial \mathbf{E}}{\partial t} \]

\[ \mathbf{\nabla} \cdot (\mathbf{\nabla} \times \mathbf{E}) = \omega \mathbf{\nabla} \cdot \mathbf{B} + \omega \frac{\partial \mathbf{E}}{\partial t} \]

Take t component
\[ \mathbf{\nabla} \cdot (\mathbf{\nabla} \times \mathbf{E}) = \frac{\partial}{\partial x} \left( \mathbf{\nabla} \times \mathbf{E} \right) = -\frac{\partial}{\partial x} \left( \mathbf{\nabla} \times (\omega \mathbf{\nabla} \times \mathbf{B}) \right) \]

Take t component
\[ \frac{\partial}{\partial x} \left( \mathbf{\nabla} \times \mathbf{E} \right) = -\frac{\partial}{\partial x} \left( \mathbf{\nabla} \times (\omega \mathbf{\nabla} \times \mathbf{B}) \right) = \mathbf{\nabla} \times (-\omega \mathbf{\nabla} \times \mathbf{B}) + \omega^{2} \left[ \mathbf{\nabla} \times \mathbf{B} \right] \]

\[ \mathbf{B}_{t}(x, y) = \frac{i \hbar}{\omega^{2}} \mathbf{\nabla} \times \mathbf{B}_{z} = \frac{i \hbar}{\omega^{2}} \mathbf{\nabla} \times \frac{\mathbf{B}_{z}}{c^{2}} \]

We need
\[ \mathbf{n} \cdot \mathbf{B} \bigg|_{s} = 0 - \mathbf{\nabla} \cdot \mathbf{B}_{t} \quad (B_{z} = 0) \]

So we have
\[ \mathbf{n} \cdot \mathbf{\nabla} \times \mathbf{B}_{z} \bigg|_{s} = 0 \]

Solutions are cosines

\[ \frac{\partial B_{2}}{\partial x} = 0 \quad \frac{\partial B_{2}}{\partial y} = 0 \]

\[ B_{z} = B_{0} \cos \frac{\pi x}{a} \cos \frac{\pi y}{b} \quad n, m, 2n, 1, 2m, \ldots \]

\[ c_{2} = \frac{\omega^{2}}{c^{2}} - \frac{n^{2} \pi^{2}}{a^{2}} - \frac{m^{2} \pi^{2}}{b^{2}} \quad \omega > \omega_{nm} \quad \text{for prop} \]
Can get other $E_x, E_y, B_x, B_y$ from Maxwell. Be given at Maxwell or boxed eqns on previous page

An important question is: what is the lowest frequency that can propagate? Cutoff $\omega_{mn}$.

The lowest mode with lowest frequency is most practical will support broader range of frequency.

$$ P_n s s - \text{given } a \neq b, a > b$$
$$ and \omega_{mn} = \sqrt{\frac{k^2 \mu \nu}{a^2 + \frac{m^2 n^2}{b^2}}} $$

and $TE_{nm}$, $TM_{nm}$ cutoff frequency which has the lowest mode which has the lowest cutoff frequency.

$\Rightarrow$ must be $TE$ because normal modes

$$ n = 1$$

$TE_{10}$
Study TE (1,0) mode

\[ W_{10} = \frac{c \pi}{a} \]

\[ \tau = \sqrt{\frac{\omega_{10}^2}{c^2} - \frac{\pi^2}{a^2}} \]

\[ B_z = B_0 \cos \frac{\pi x}{a} e^{i (k z - \omega t)} \]

\[ B_x = \frac{1}{\tau} \frac{\partial B_z}{\partial y} \cdot \frac{\omega^2}{\omega^2 - \omega_{10}^2} \cdot \frac{\pi}{a} \left( -\sin \frac{\pi x}{a} \right) e^{i (k z - \omega t)} \]

\[ \tau = \frac{\omega}{c} \]

\[ B_y = 0 \]

\[ \vec{E} = -\vec{E}_0 \cdot \nabla \times \vec{B}_z \]

\[ B_x = 0 \]

\[ E_y = \omega \sigma \left( \frac{\pi}{a} \right) \frac{B_0}{\pi} \sin \frac{\pi x}{a} e^{i (k z - \omega t)} \]

\[ S = \frac{\sigma}{\rho_0} \vec{E} \cdot \vec{B} \]

\[ \frac{\vec{B} \times \vec{E}}{\rho_0} \]
We are interested in how fast waves propagate.

Take TE: \[ B_2 = B_0 \cos \frac{n \pi x}{a} \cos \frac{m \pi y}{b} \]
\[ e^{i(kz - \omega t)} \]

with \[ \frac{k^2}{\varepsilon^2} = \frac{\omega^2}{c^2} - \frac{n^2 \pi^2}{a^2} - \frac{m^2 \pi^2}{b^2} = \frac{1}{\varepsilon^2} \left( \omega^2 - \omega_{nm}^2 \right) \]

There are two relevant velocities:

Without BC we had \[ \frac{k}{\varepsilon} = \frac{\omega}{c} \]

\[ c = \text{wave velocity in free space} \]
\[ \text{in dielectric} \]
\[ N = \frac{\omega}{k} = \frac{c}{n} \]

Here we distinguish 2 kinds of velocities:

Phase velocity (comes from the phase) \( v_{ph} \)
\[ v_{ph} = \frac{\omega}{k} = c \frac{\omega}{\sqrt{\omega^2 - \omega_{nm}^2}} > c !!! \]

Another velocity is the group velocity
\[ v_g = \frac{d\omega}{dk} \]

\[ (1) \Rightarrow c \frac{dv}{dh} = \omega d\omega \Rightarrow \frac{d\omega}{dh} = c \frac{\lambda}{\omega} \]
\[ \frac{\partial \mathbf{u}}{\partial t} = c \sqrt{1 - \frac{\beta^2}{c^2}} \frac{\mathbf{u} - \beta \mathbf{e}_x}{b^2} \quad \text{< c} \]

The wave actually propagates with speed \( \frac{\partial \mathbf{u}}{\partial t} \). True in AM too. Why?

\[ \text{Energy/area/area} = \langle \mathbf{S} \rangle \quad \mathbf{S} = \mathbf{e} \times \mathbf{B} \]

Integrate over area to get energy per time

\[ <W> = \text{Energy / volume} \]

Integrate over area to get energy/length

\[ = \langle \mathcal{S} \mathbf{a} \cdot u \rangle \]

\[ \text{Speed} = \frac{\int \mathcal{S} \rho <\mathbf{e}^2>}{\int \mathcal{S} \rho <u^2>} = \frac{\partial \mathbf{u}}{\partial t} \]

\[ = \frac{\text{Energy / time}}{\text{Energy / length}} \]
E 

Resonant Cavity

Suppose enclose both ends of a wave guide with conductor (need small hole to inject energy)

Microwave cavity acts as a resonant circuit

Solutions of Maxwell's standing waves: sines and cosines instead of 
Electric field

Special only waves that can exist satisfy

Boundary conditions at 6 walls. Special set of frequencies:

Resonant frequencies.

Entrapment of energy at discrete f & enable use of cavity to measure frequencies

Can have movable wall as a tuner; cavity is similar to organ pipe or sound box

In musical instrument. Cavities are used in oscillators and transmitters to create microwave
Signals, filters to separate signals at a given frequency from other signals. Use in radar, microwave relays, satellite communication, & microwave ovens.

**Theory**

**TE Mode**

\[ B_z = B_0 \cos \frac{2\pi a}{a} \cos \frac{2\pi y}{b} e^{i(k_z - \omega t)} \]

Need \pm because waves reflect off surfaces at \( z = 0, d \)

on surfaces \( z = 0, d \) \( \vec{A} \cdot \vec{B} = 0 \rightarrow B_z = 0 \)

\( c \frac{\partial B_z}{\partial t} = \sin \frac{2\pi z}{d} = 0 \) at \( z = 0 \)

to vanish at \( z = d \) \( k_z = \frac{2\pi p}{d} \) \( p = 1, 2, 3, \ldots \)

Wave eqn \( \frac{\partial^2 A}{\partial t^2} + \frac{\omega^2}{c^2} - k_z^2 = 0 \)

\[ \frac{\omega}{c} = \sqrt{\left(\frac{n}{a}\right)^2 + \left(\frac{m}{b}\right)^2 + \left(\frac{p}{d}\right)^2} = \frac{\omega_{nmp}}{c} \]

Mode defined by \( nmp \) \( \text{TE}_{nmp} \)

In real cavities \( \omega \) does not have to be exactly \( \omega_{nmp} \). There are losses of energies in conductors that allow frequencies to have a spread — not in this course.