April 9

Micro waves are important in current elec. eng.

$300\,MHz < \nu < 300\,GHz$

$1mm < \lambda < 1m$

ability to generate μwaves & control intensity →

new technologies used everyday.

First use - radar

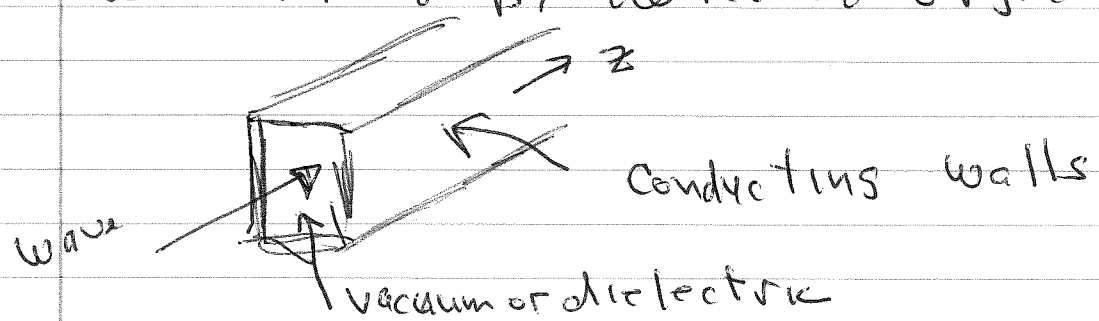
- WW II military, now
- Air Traffic Control
- Obs Weather
- Speed limits

Basic idea: detect object by scattering of waves - $\lambda \sim$ sizes of object

More uses

- Cell phone system (1981) ν 824-849 MHz & 869-894 "
- μwave links between antennas carry TV & telephone (470-890 MHz)
- μwave oven 2.45 GHz
- Cosmic scale - radiation remnant of big bang is black body spectrum at T = 3K, max in μwave range

Wave guides (wg) are an essential component of micro wave circuits - hollow tube in which EM waves propagate with little attenuation. Radar μwaves generated in a resonant cavity travel thru wg to antenna from which emerge in free space to be scattered by detected objects



Want to study how waves propagate.

Complication - fields make electrons in walls (boundaries) oscillate and extra waves are generated

Radiation sent into wg reflects from any conducting wall that it hits so that it bounces back & forth

between opposite walls. Fields in guides & walls influence each other. Math - PDE Maxwell eq w

Boundary conditions.

The math is similar to quantum mechanics - QM

only discrete modes can propagate

E & M 6 quantities $\vec{E}$, & $\vec{B}$
QM ψ

The walls will be assumed to be

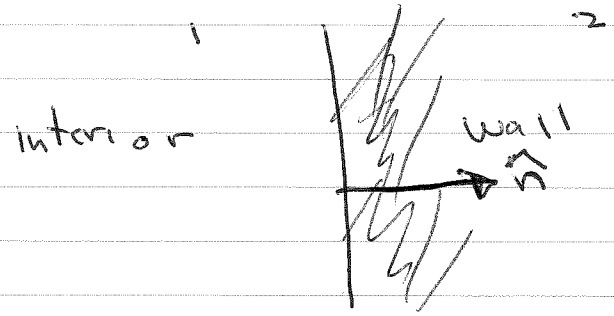
perfect conductors $\sigma \to \infty$ conductivity
skin depth is 0 E & B don't penetrate
metal. Only on surface

Logic

$\vec{E} = 0$ inside perfect conductor
Faraday $\partial \vec{B}/\partial t = 0$, assume $\vec{B} = 0$ initially
stays 0

$\boxed{\text{Goal - compute } \vec{E} \text{ \& } \vec{B} \text{ that propagate}}$

Boundary value problem



Boundary conditions AT SURFACE

Tangential $\vec{E}$ is continuous

$$\hat{n} \times (\vec{E}_1 - \vec{E}_2) = 0 \qquad E_2 = 0$$

$$\hat{n} \times \vec{E}_1 = 0$$

Components of $\vec{E}$ $\parallel$ to surface $= 0$

normal compo of B is continuous $\qquad \vec{\nabla} \cdot \vec{B} = 0$

$$\hat{n} \cdot (\vec{B}_1 - \vec{B}_2) = 0 \rightarrow \hat{n} \cdot \vec{B}_1 = 0$$

Comp of $\vec{B} \perp$ to surface $= 0$

So to summarize $\quad \vec{E}_{\parallel} = 0 \quad \vec{B}_{\perp} = 0$ at surface.

Have <u>hollow</u> conductor - propagation in z direction Assume harmonic time dependence

$$\vec{E}(x, y, z, t) = \vec{E}(x, y, z)\, e^{-i\omega t}$$

$$\vec{B}(x, y, z, t) = \vec{B}(\vec{r})\, e^{-i\omega t}$$

Maxwell $\quad \vec{\nabla} \cdot \vec{E} = 0 \quad \vec{\nabla} \cdot \vec{B} = 0$

$$\vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} = i\omega \vec{B} \qquad \vec{\nabla} \times \vec{B} = \frac{1}{c^2}\frac{\partial \vec{E}}{\partial t} = -\frac{\omega^2}{c^2} i \vec{E}$$

If medium is not hollow $\quad \frac{1}{c^2} \rightarrow \frac{\mu\epsilon}{\mu_0 \epsilon_0} = \frac{1}{v^2} \qquad \mu = \mu_0$

Take $\vec{\nabla} \times (\vec{\nabla} \times \vec{E})$ as $\nabla \times B \rightarrow$ wave eq

$$\left(\nabla^2 + \frac{\omega^2}{c^2}\right)\left\{\begin{matrix}\vec{E}\\ \vec{B}\end{matrix}\right\} = 0$$

Assume wave guide shape is

<u>same</u> for all values of z

$\vec{E} = \vec{E}_z + \vec{E}_t$ $\qquad$ t = transverse to prop direction

$\vec{E}_z = \hat{z}\cdot\vec{E}$ $\qquad$ Same notation for $\vec{B}$

Want waves to propagate in z-direction

$$\vec{E}(x,y,z) = \vec{E}(x,y)\, e^{ikz}$$
$$\vec{B}(x,y,z) = \vec{B}(x,y)\, e^{ikz}$$

$$\nabla^2 \rightarrow \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} - k^2 = \nabla_t^2 - k^2$$

so $$\left(\nabla_t^2 + \frac{\omega^2}{c^2} - k^2\right)\left\{\begin{matrix}\vec{E}(x,y)\\ \vec{B}(x,y)\end{matrix}\right\} = 0$$

What kind of waves can propagate

1) TE $\quad$ Transverse Electric $\quad$ $E_z = 0$
2) TM $\quad$ " $\quad$ Magnetic $\quad$ $B_z = 0$

General wave sums of both kinds

3) TEM, $E_z = 0$ $B_z = 0$?

This happens in free space, now situation is different because of BC

$E_z = 0, \; B_z = 0$?

We have $\vec{\nabla} \cdot \vec{E} = 0$

with $B_z = 0$ Take z-component of $\vec{\nabla} \times \vec{E} = \frac{\omega}{c} i \vec{B}$

so z-comp. of $\vec{\nabla} \times \vec{E} = 0$

so $\vec{\nabla} \times \vec{E}_t = 0$

with $E_z = 0$ $\vec{\nabla} \times \vec{E} = 0$

Both $\vec{\nabla} \cdot \vec{E}$ & $\vec{\nabla} \times \vec{E} = 0$ so

$\vec{E}$ must vanish no TEM field at all

Note if there is a separate conductor in the middle

there will be TEM mode. For now have TE & TM

Goal compute allowed solutions modes that solve BC & ω Eq

<u>Summary</u> $\vec{E}_{\vec{B}}(x,y)\, e^{ikz - i\omega t}$

TE $E_z = 0$ TM $B_z = 0$

Surface Boundary conditions 1) $E_{\parallel} = 0$ $E_z = 0$ 2) $\hat{n} \cdot \vec{B} = 0$

$B_t = 0$

$$\left(\nabla_\perp^2 + \frac{\omega^2}{c^2} - k^2\right) \begin{Bmatrix} E \\ B \end{Bmatrix} = 0$$

Solution strategy

TE: $B_z = 0$ Solve wave eq for B_z

Use Maxwell to get E_x, E_y, B_x, B_y from B_z

TM: $B_z = 0$ Solve wave eq for E_z (why important)

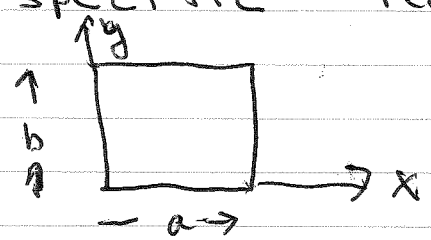
Use Maxwell to get E_x, E_y, B_x, B_y from E_z

Do TM first

$$\left(\nabla_t^2 + \frac{\omega^2}{c^2} - k^2\right) E_z = 0 \longrightarrow \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\omega^2}{c^2} - k^2\right) E_z = 0 \quad (1)$$

with $E_z = 0$ on surface

To be specific rectangular wave guide



$$E_z = X(x)\, Y(y)$$

with $X(x=0,a) = 0$, $Y(y=0,a) = 0$

$$Y \frac{d^2 X}{dx^2} + X \frac{d^2 Y}{dy^2} = \left(\frac{\omega^2}{c^2} + k^2\right) XY$$

Divide by XY

$$\frac{1}{X} X'' + \frac{1}{Y} Y'' = k^2 - \frac{\omega^2}{c^2} = \text{const}$$

Each of $\frac{X''}{X}, \frac{Y''}{Y}$ must be constant

the constant must be negative

because the BC repeat

$$X = \sin k_1 x \quad , \quad \frac{X''}{X} = -k_1^2$$

with $k_1 a = n\pi$ $\qquad$ <u>not all k_1 work</u>

$$Y = \sin k_2 y \quad , \quad \frac{Y''}{Y} = -k_2^2$$

$k_2 b = m\pi$

and from (1) $\quad -k_1^2 - k_2^2 = k^2 - \frac{\omega^2}{c^2}$

$$k^2 = \frac{\omega^2}{c^2} - \left(\frac{n\pi}{a}\right)^2 - \left(\frac{m\pi}{b}\right)^2$$

k^2 must be > 0, from e^{ikz} factor, k real for prop

A <u>mode</u> is defined by the numbers n, m

For a given n, m

$$\omega > c\sqrt{\left(\frac{n\pi}{a}\right)^2 + \left(\frac{m\pi}{a}\right)^2} \equiv \omega_{n,m}$$

$\omega_{n,m}$ is cutoff frequency of mode

~~Now~~ Thus $\underline{E_z = E_0 \sin\frac{n\pi}{a}x \sin\frac{m\pi}{a}y\, e^{ik_z(\omega,n,m)z}}$

get other components

How? Maxwell eq

$$\vec{\nabla} \times \vec{B} = -\frac{i\omega}{c^2} \vec{E}$$

$$\vec{\nabla} \times \vec{E} = i\omega \vec{B}$$

we need to separate transverse and $\hat{z}$ components

Take $\hat{z} \times$ these eqns

Use vector identity any vector $\vec{V}$

$$\hat{z} \times (\vec{\nabla} \times \vec{V}) = \vec{\nabla}(\vec{V} \cdot \hat{z}) - \frac{\partial}{\partial z} \vec{V}$$

Find $\vec{B}_t$

$$\hat{z} \times (\vec{\nabla} \times B) = \vec{\nabla} B_z - \frac{\partial \vec{B}}{\partial z} = -\frac{i\omega}{c^2} \hat{z} \times \vec{E}$$

TM 0 $\qquad -ik\vec{B}_t$

$$\vec{B}_t = \frac{\omega}{kc^2} \hat{z} \times \vec{E}_t$$

Find E_t $\qquad \hat{z} \times (\vec{\nabla} \times \vec{E}) = i\omega \hat{z} \times \vec{B}$

$$\vec{\nabla} E_z - ik\vec{E} = i\omega \left(\hat{z} \times \frac{\omega}{kc^2}(\hat{z} \times \vec{E}_t)\right) = \frac{i\omega^2}{kc^2}(-\vec{E}_t)$$

Known

A vector equation. z component is $0 = 0$ Physics is not.

Transverse component

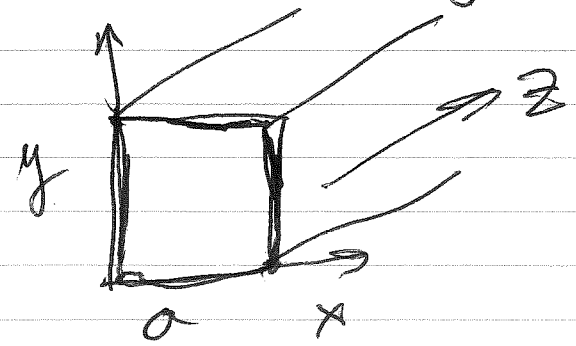
$$\vec{E}_t = \frac{ik}{\frac{\omega^2}{c^2} - k^2} \vec{\nabla}_t E_z(x,y) = \frac{ik}{\frac{\omega_{n,m}^2}{c^2}} \vec{\nabla}_t E_z$$

$$\vec{\nabla}_t = \hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y}$$

Here

4/11

Last time wave guides



Summary

0 conductivity → ∞

Waves propagate as $e^{i(kz-\omega t)}$

Modes TM yesterday $B_z = 0$

Ideas Maxwell's eq → wave eq

$$\left(\nabla_t^2 + \frac{\omega^2}{c^2} - k^2\right)\begin{Bmatrix}\vec{E}\\ \vec{B}\end{Bmatrix} = 0$$

$$\nabla_t^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$$

BC $E_{\parallel}$ to surface $= 0$
$B_{\perp}$ to surface $= 0$

Found solns called modes
Get $E_{x,y}$ B_x, B_y from E_z

Today continue with more interesting case

TE $E_z = 0$ $c^2k^2 = \omega^2 - \omega_{n,m}^2$

$$\omega_{n,m} = \sqrt{\left(\frac{\pi n}{a}\right)^2 + \left(\frac{\pi m}{b}\right)^2}$$ cut off

TE modes $E_z = 0$ First get B_z

This is more difficult than TM

because $\hat{n} \cdot \vec{B}\big|_S = 0$

$B_t = 0$ on surface must get B_t in terms of B_z.

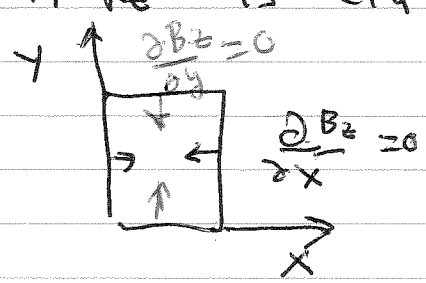
There are two main BC's in UG physics

function $= 0$ or function' $= 0$

It is natural to expect that
$\hat{n} \cdot \vec{\nabla}_t B_z = 0$ on surface

This is because all solutions are Linear combos of TE & TM and TM uses $E_z\big|_S = 0$

$\hat{n} \cdot \vec{\nabla}_t$ is either $\frac{\partial}{\partial x}$ or $\frac{\partial}{\partial y}$



The actual BC is that $\hat{n} \cdot \vec{B}\big|_S = 0 = \hat{n} \cdot \vec{B}_t\big|_S$

Need to get condition on B_z from this

$$\vec{\nabla}\times\vec{E} = i\omega\vec{B} \qquad \vec{\nabla}\times\vec{B} = -\frac{i\omega}{c^2}\vec{E}$$

$$\hat{z}\times(\vec{\nabla}\times\vec{E}) = i\omega\,\hat{z}\times\vec{B}_t \qquad \text{take } \vec{t} \text{ component}$$

$$\cancel{\vec{\nabla}_t(E_z)} - \frac{\partial}{\partial z}\vec{E}_t$$

$$-ik\vec{E}_t = i\omega\,\hat{z}\times\vec{B}_t, \qquad \boxed{\vec{E}_t = -\frac{\omega}{k}\hat{z}\times\vec{B}_t} \checkmark$$

$$\hat{z}\times(\vec{\nabla}\times\vec{B}) = -\frac{i\omega}{c^2}\hat{z}\times\vec{E}_t \qquad \text{take } \vec{t} \text{ com}$$

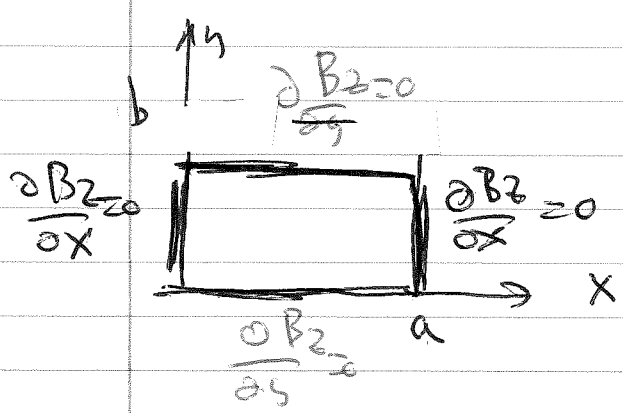
$$= \vec{\nabla}B_z - ikB = -\frac{i\omega}{c^2}(-\omega/k)\,\hat{z}\times(\hat{z}\times\vec{B})$$

$$\vec{\nabla}_t B_z - ikB_t = -\frac{i\omega}{c^2}\hat{z}\times\left(-\frac{\omega}{k}\hat{z}\times\vec{B}_t\right) = +\frac{i\omega^2}{c^2k}\left[-\vec{B}_t\right]$$

$$\rightarrow \boxed{\vec{B}_t(x,y) = \frac{ik}{\frac{\omega^2}{c^2}-k^2}\vec{\nabla}_t B_z}$$

we need $\hat{n}\cdot\vec{B}\big|_S = 0 = \hat{n}\cdot\vec{B}_t\big|_S \qquad (B_z = 0)$

So BC is $\hat{n}\cdot\vec{\nabla}_t B_z\big|_S = 0$



Solutions are cosines

$$B_z = B_0\cos\frac{n\pi}{a}\cos\frac{m\pi}{b} \qquad n,m = 0,1,2,3\ldots$$

$$k^2 = \frac{\omega^2}{c^2} - \frac{n^2\pi^2}{a^2} - \frac{m^2\pi^2}{b^2} \qquad \omega > \omega_{nm} \text{ for prop}$$

Can get other ~~eqns~~ E_x, E_y, B_x, B_y from ~~Maxwell~~ ~~B_z given a b~~ Maxwell or boxed eqns on previous page

An important question is what is the lowest frequency that can propagate?

The ~~lowest~~ mode with lowest cutoff frequency $\omega_{n,m}$ is most practical will support broader range of frequency

PNSS - given a & b , $a > b$

and $\omega_{nm} = \sqrt{\frac{n^2\pi^2}{a^2} + \frac{m^2\pi^2}{b^2}}$

and $TE_{n,m}$, $TM_{n,m}$ mode which has the lowest cutoff frequency

→ must be TE because n or m can be 0

sent $m = 0$
$n = 1$

TE_{10}

Study TE (1,0) mode

$$\omega_{1,0} = \frac{c\pi}{a}$$

$$k = \sqrt{\frac{\omega^2}{c^2} - \frac{\pi^2}{a^2}}$$

$$B_z = B_0 \cos\frac{\pi x}{a} \; e^{i(kz-\omega t)}$$

$$B_x = \frac{ik}{\frac{\omega^2}{c^2} - k^2} \frac{\partial B_z}{\partial x} = \frac{ik}{\omega_{1,0}^2}\left(-\sin\frac{\pi x}{a}\right)\frac{\pi}{a} B_0 e^{i(kz-\omega t)}$$

$$= -\frac{ika}{\pi} B_0 \sin\frac{\pi x}{a} \; e^{i(kz-\omega t)}$$

$$\vec{B}_t = B_x \hat{x}$$

$$B_y = 0$$

$$\vec{E} = -\frac{\omega}{k} \hat{z} \times \vec{B}_t \quad , \quad \hat{z} \times \hat{x} = \hat{y}$$

$$E_x = 0$$

$$E_y = \frac{\omega}{k}\left(+i\frac{ka}{\pi}\right) B_0 \sin\frac{\pi x}{a} \; e^{i(kz-\omega t)}$$

$$\vec{S} = \frac{E_x B_y}{\mu_0} \hat{x} \times \hat{y}$$

$$= \frac{E_x B_y}{\mu_0} \hat{z}$$

We are interested in how fast waves propagate

Take TE $\quad B_z = B_0 \cos \frac{n\pi}{a} x \cos \frac{m\pi}{b} y \, e^{i(kz - \omega t)}$

with $\quad k^2 = \frac{\omega^2}{c^2} - \frac{n^2\pi^2}{a^2} - \frac{m^2\pi^2}{b^2} = \frac{1}{c^2}\left(\omega^2 - \omega_{n,m}^2\right) \quad (1)$

There are two relevant velocities

without BC we had $k = \frac{\omega}{c}$

c = wave velocity in free space

in dielectric

wave velocity $\quad v = \frac{\omega}{k} = \frac{c}{n}$

Here we distinguish 2 kinds of velocities

phase velocity (comes from the phase) v_{ph}

$$v_{ph} = \frac{\omega}{k} = c \frac{\omega}{\sqrt{\omega^2 - \omega_{n,m}^2}} > c \; !!$$

Another velocity is the group velocity

$$v_g = \frac{d\omega}{dk} =$$

$(1) \Rightarrow c^2 k \, dk = \omega \, d\omega \rightarrow d\omega/dk = c^2 \frac{k}{\omega}$

$$v_g = c\sqrt{1 - \frac{n^2\pi^2}{a^2} - \frac{m^2\pi^2}{b^2}} < c$$

The wave actually propagates with speed v_g! True in QM too

Why?

Energy/unit area/unit time $= \langle \vec{S} \rangle$ time ave $\quad \vec{S} = \vec{E} \times \vec{H}$

Integrate over area to $\frac{\text{energy}}{\text{time}} = \langle \int da\, \vec{S} \rangle$

$\langle u \rangle =$ Energy / volume $\quad u = \frac{1}{2}(\epsilon E^2 + \mu B^2)$

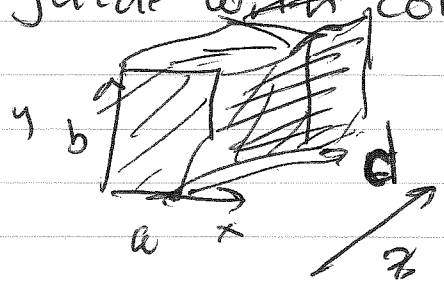
Integrate over area to get energy/length $= \langle \int da\, u \rangle$

$$\text{speed} = \frac{\int da \langle \vec{S} \rangle}{\int da \langle u \rangle} = v_g$$

$= \frac{\text{energy/time}}{\text{energy/length}}$

Resonant μwave cavity

suppose enclose both ends of wave guide ~~with~~ conductor



(need small hole to inject energy)

μ wave cavity acts as a resonant circuit

Solutions of maxwell eqns standing waves
sines cos insted of e^{ikz}, waves bounce back and forth

Special the only waves that can exist satisfy bc at 6 walls. Special set of frequencies resonant frequencies.

Entrapment of energy at discret ~~E~~ enabler

use of cavity to measure frequencies

Can have movable wall as a tuner Cavity is similar to organ pipe, or sound box in musical instrument. Cavities are used in oscillators and transmitters to create μwave

signals, filters to separate signals at a given frequency from other signals. Use in radar, μwave relays, satellite communications & μwave ovens

Theory

TE Mode $B_z = B_0 \cos\frac{n\pi x}{a} \cos\frac{m\pi y}{b} e^{\pm ikz - i\omega t}$

Need $\pm$ because waves reflect off surfaces at $z = 0, d$

On surfaces $z=0,d$ $\hat{n}\cdot\vec{B} = 0 \rightarrow B_z = 0$

$e^{\pm ikz} \rightarrow \sin kz$ $= 0$ at $z=0$
to vanish at $z = d$ $k = \frac{p\pi}{d}$ $p = 1, 2, 3, \ldots$

Wave eqn $\nabla_t^2 + \frac{\omega^2}{c^2} - k^2 = 0$

$$\rightarrow \frac{\omega}{c} = \sqrt{\left(\frac{n\pi}{a}\right)^2 + \left(\frac{m\pi}{b}\right)^2 + \left(\frac{p\pi}{d}\right)^2} = \frac{\omega_{nmp}}{c}$$

Mode defined by $n\ m\ p$ TE_{nmp}

In real cavities ω does not have to be exactly ω_{nmp} - There are losses of energies in conductors that allow frequencies to have a spread — subject not in this course