NAME: SOLUTIONS **Student ID:** **Score:**

Physics 323 **Spring 2017**

EXAM # 1
9 AM - 1020 AM, Thursday April 20

Write your name and ID number at the top of this page and on pages 2-5.

Clearly show all your reasoning.

You may use a calculator, but computers or other programmable devices are not allowed.

You are not allowed to use your phone during the exam.

This is a closed-book exam. Textbooks, class notes and other class material are not allowed.

Relevant formulae and equations are provided in a separate booklet.

Clearly note all constants and assumptions you use.

Show all your work and your final answers in the spaces provided. If you need to use the back of a page to complete your answer, clearly indicate this. Scratch work will not be graded.

Extra paper is available at the front of the classroom.

If you have a question during the exam, raise your hand.

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1. (25 pts total) *Rectangular Waveguide*
Electromagnetic waves of frequency $\omega = 2\pi \times 10\,\text{GHz}$ enter a hollow rectangular waveguide with inside transverse dimensions 2.4 cm and 1 cm.
(a) (8 pts) Which if any are the propagating TE modes?

The cutoff frequencies for TE_{mn} modes are $\nu_{mn} = \frac{\omega_{mn}}{2\pi} = \frac{c}{2}\sqrt{(\frac{m}{a})^2 + (\frac{n}{b})^2}$ with the lowest allowed being TE_{10}. Here $a = 2.4$ cm, $b = 1.0$ cm and $c = 3 \times 10^{10}$ cm/s. We determine the first few cutoff frequencies to be $\nu_{10} = \frac{3\times10^{10}}{2\times2.4} = 6.3$ GHz, $\nu_{20} = 2\nu_{10} = 12.6$ GHz. $\nu_{01} = \frac{3\times10^{10}}{2\times1.0} = 15$ GHz, $\nu_{11} = \frac{3\times10^{10}}{2}\sqrt{(\frac{1}{2.4})^2 + (\frac{1}{1.0})^2}, = 16.3$ GHz, $\nu_{21} = \frac{3\times10^{10}}{2}\sqrt{(\frac{2}{2.4})^2 + (\frac{1}{1.0})^2}, = 19.5$ GHz
Only $\nu_{10} < \frac{\omega}{2\pi}$, thus only the TE_{10} mode propagates.

(b) (5 pts) What range of entering frequencies would propagate in only one TE mode? Which mode is this?

The range of entering frequencies correspond to the frequencies above the lowest cutoff frequency and below the second lowest cutoff frequency. From (a), these correspond to ν_{10} and ν_{20}. Thus the range of entering frequencies that satisfy this criterion is the range between 6.3 GHz and 12.6 GHz (multiply by 2π for angular frequency). The solitary TE mode that will propagate is the TE_{10} mode.

(c) (5 pts) What range of entering frequencies would propagate in only one TM mode? Which mode is this?

For TM modes neither m nor n is allowed to be zero in the expression for ν_{mn} in (a). Thus the lowest allowed mode is TM_{11} and the required range of frequencies corresponds to between ν_{11} and ν_{21}, ie between 16.3 GHz and 19.5 GHz. The solitary TM mode that will propagate is the TM_{11} mode.

(d) (7 pts) Suppose the waveguide is now filled with a lossless (non-conducting) linear dielectric having index of refraction $n = 2$. Now, which if any are the propagating TE modes? Explain.

In this case, the speed c needs to be replaced by $c/n = c/2$ and the cutoff frequencies ν_{mn} are each lowered by a factor of 2. All the modes considered in (a) will now have cutoff frequencies $\nu_{mn} < 10$ GHz, and thus will be allowed.
Consider additional modes: $\nu_{30} = 3 \times 6.3/2 = 9.5$ GHz, $\nu_{40} = 4 \times 6.3/2 = 12.6$ GHz, $\nu_{02} = 2 \times 15/2 = 15$ GHz, $\nu_{31} = \frac{3\times10^{10}}{4}\sqrt{(\frac{3}{2.4})^2 + (\frac{1}{1.0})^2} = 12.0$ GHz, $\nu_{22} = \frac{3\times10^{10}}{4}\sqrt{(\frac{2}{2.4})^2 + (\frac{2}{1.0})^2} = 16.3$ GHz. We find $\nu_{30} < 10$ GHz. Thus the propagating TE modes are TE_{10}, TE_{20}, TE_{30}, TE_{01}, TE_{11}, TE_{21}.

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2. (25 pts total) *Gold Layer*
The conductivity σ of gold is around $10^8\,(\Omega\text{m})^{-1}$, satisfying $\sigma \gg \epsilon\omega$ for frequencies well above visible light. Suppose you want to plate a thin layer of gold on a section of poor conductor in order to have a good conductor at frequency $\omega = 2\pi \times 10\,\text{GHz}$.
(a) (7 pts) What is the order-of-magnitude estimate for how thick this gold layer needs to be?

The gold layer needs to be thicker than the skip depth so that the radiation does not escape into the poor conductor. Gold is a good conductor at 10 GHz, with $\sigma \gg \epsilon\omega$. Thus skin depth $d = 1/\kappa$ where $\kappa =\text{Im}(\tilde{k}) = \sqrt{\frac{\omega\sigma\mu}{2}}$, for a good conductor.
Gold is non-magnetic, so $\mu = \mu_0$, and we get $d =\simeq 5\times 10^{-7}$ m or about 500 nm. Thus, the gold layer needs to be (at least) 500 nm thick.

(b) (6 pts) What is the wavelength of this 10 GHz electromagnetic wave in the gold layer?

For a good conductor, wavelength $\lambda_{\text{gold}} = \frac{2\pi}{\text{Re}(\tilde{k})} = 2\pi\sqrt{\frac{2}{\omega\sigma\mu}} \simeq 2\pi(5\times 10^{-7}) \simeq 3\times 10^{-6}$ m.
So the wavelength of this radiation in gold is about 3μm.

(c) (6 pts) What is the speed of this 10 GHz electromagnetic wave in the gold layer?

Wave speed in the gold layer $= \omega/\text{Re}(\tilde{k}) = \lambda_{\text{gold}}(\frac{\omega}{2\pi}) \simeq 3\times 10^{-6}(10^{10}) = 3\times 10^{4}$ m/s.

(d) (6 pts) What is the order-of-magnitude estimate for the skin-depth for visible light in the gold layer?

We can take the angular frequency for visible light ω_{vis} to be 10^{15}/s. Again using the expression for skin depth for a good conductor, we get $d \simeq \sqrt{\frac{2}{(2\pi\times 10^{15})(10^{8})(4\pi\times 10^{-7})}} \simeq 1.6\times 10^{-9}$ or about 1.6 nm.

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3. (25 pts total) *Dilute Dispersive Gaseous Medium*
A *dilute* gaseous medium has a single optical resonance at frequency ω_0. The intensity of a plane wave at frequency ω_0 propagating through the gas is attenuated by a factor of two over a distance of 10 meters. The frequency width of the absorption resonance is $\Delta\omega$.
(a) (6 pts) What is the distance over which the electric field associated with the wave gets reduced by a factor of two?

The electric field gets attenuated as $e^{-\kappa z}$ and since the intensity is proportional to the square of the electric field, the intensity gets attenuated as $e^{-2\kappa z}$.
The distance D over which the electric field is attenuated by a factor of 2 satisfies $e^{-\kappa D} = 1/2$ and we know $e^{-2\kappa(10)} = 1/2 = e^{-20\kappa}$. Thus $D = 20$ meters.

(b) (7 pts) Arrange in ascending order the propagation (phase) velocities at frequencies ω_0, $\omega_0 + \Delta\omega/10$, and $\omega_0 - \Delta\omega/10$. Show your reasoning. Hint: How does n vary with ω?

Within the resonance linewidth, we have anomalous dispersion and n decreases with ω. The phase velocity $v_p = c/n$.
Thus, $v_{p,\omega_0-\Delta\omega/10} < v_{p,\omega_0} < v_{p,\omega_0+\Delta\omega/10}$.

(c) (6 pts) Using the relationship $k = n\omega/c$ where n is the (real) index of refraction and k is the real part of $\tilde{k}$, show that the group velocity can be written as $v_g = \frac{c}{n+\omega\frac{dn}{d\omega}}$.

By definition, $v_g = \frac{1}{dk/d\omega}$. Using the relation between $k = \frac{n\omega}{c}$, we get $\frac{dk}{d\omega} = \frac{1}{c}(n + \omega\frac{dn}{d\omega})$.
Therefore, $v_g = \frac{1}{dk/d\omega} = \frac{c}{n+\omega\frac{dn}{d\omega}}$.

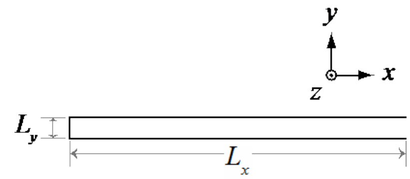
(d) (6 pts) Using the relation derived in (c), determine whether the group velocity is larger or smaller than the phase velocity at ω_0. Explain.

Around resonance we have anomalous dispersion and $\frac{dn}{d\omega} < 0$. Thus, $v_g > c/n$ and the group velocity is larger than the phase velocity at ω_0.

Name ____________________________________ **Student ID** ______________ **Score**________
last *first*

IV. [25 points total] Tutorial question.

Consider a waveguide made from a hollow conducting rectangular pipe whose height L_y is *much smaller* than its width L_x, and is very long in the z-direction.



Cross-sectional view

A. [7 pts] Determine the first three modes (*i.e.* mode *x, y*) for this waveguide. If there is not enough information, state so explicitly. Explain your reasoning.

The first three modes would be 1,0; 2,0; and 3,0.

The cutoff frequency of a mode is based on the geometry of the waveguide, involving terms that look like $\left(\frac{n_i}{L_i}\right)^2$. *With the approximation* L_y *is much smaller than* L_x*, the 0,1 mode will be larger than any* n_x*,0 mode.*

B. [8 pts] For a single frequency of light 3.5 times the lowest cutoff frequency for the waveguide, qualitatively describe the angle(s) or range of angles at which light can leave the waveguide, if light can enter the waveguide at any angle (0 angle is down the waveguide, 90° or π/2 is across the waveguide). Explain your reasoning.

Light of this frequency can leave this waveguide at discrete angles (specifically, 3 discrete angles).

One frequency of light corresponds to one value for the magnitude of $\vec{k}$. *A given mode defines the perpendicular components of* $\vec{k}$. *With a geometric model of light bouncing within the waveguide, a given frequency of light travelling at a given mode can only travel at one discrete angle.*

N.b., you can find these angles by trigonometry with the $\vec{k}$ *triangle. With the geometry of the waveguide, the perpendicular component is either 1, 2, or 3 units long, so the angles are arcsin(1/3.5, 2/3.5, or 3/3.5).*

C. [10 pts] Consider two pulses of light centered on two different frequencies travelling through the waveguide in the same mode (both frequencies are above the mode's cutoff frequency). The two lights pulse at the same time on one end of the waveguide. Does light from the pulse centered on the higher frequency appear at the other end of the waveguide *before, after,* or *at the same time as* that of the lower frequency? Explain your reasoning.

The pulse with the higher frequency reaches the other end before the pulse with the lower frequency.

Geometric model: A higher frequency corresponds to a bigger magnitude of $\vec{k}$ *by* $c = {}^{\omega}/_{k}$. *A given mode determines the perpendicular component of* $\vec{k}$*, so thus the larger magnitude of* $\vec{k}$ *travels down the waveguide at a shallower angle (closer to the z-axis). Light travels in vacuum the same, but the shallower angle means a larger z-component of the velocity. Thus, the higher frequency reaches the end first (by* $\Delta x = v\Delta t$*).*

Group velocity: You can apply the group velocity definition of $v_g = \frac{1}{dk/d\omega}$ *the guided wavenumber equation* ${k_g}^2 = \left(\frac{\omega}{c}\right)^2 - \left(\frac{\omega_{\text{cutoff}}}{c}\right)^2$ *and get that the group velocity is* $v_g = c^2\sqrt{\left(\frac{\omega}{c}\right)^2 - \left(\frac{\omega_{\text{cutoff}}}{c}\right)^2}/\omega$. *Then you can compare higher values of ω to lower values of ω (or compare the derivative of the numerator v. the derivative of the denominator) to see that the group velocity is higher if ω is higher.*