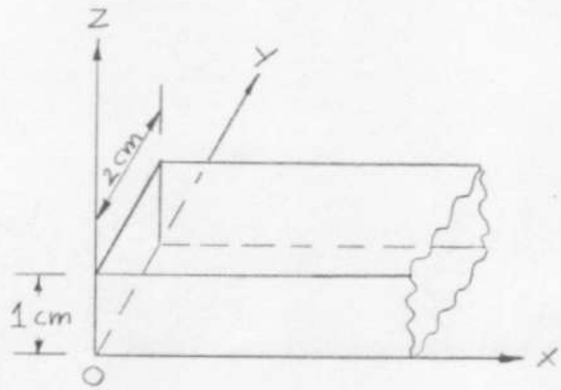
Name ______________________________ **Student ID** ____________ **Score** ______
last *first*

I. Rectangular waveguide. Consider the rectangular waveguide as shown. The guided axis is along the x-direction, with height (z-direction) 1cm and width (y-direction) 2 cm. The volume inside is vacuum and the walls are perfect conductors. A wave propagates down the guide at angular frequency ω.



a. (5 pts) Write the boundary conditions, if any, on E_x, E_y, E_z, B_x, B_y and B_z at the walls.
$\parallel E$ AND $\perp B$ VANISH AT SURFACE. HENCE:
(Y=0 & 2 CM) $B_Y=0,\ E_X=E_Z=0;\ \partial E_Y/\partial Y=0$ (FROM $\vec{\nabla}\cdot\vec{E}=0$)
(Z=0 & 1 CM) $B_Z=0,\ E_X=E_Y=0;\ \partial E_Z/\partial Z=0$ (FROM $\vec{\nabla}\cdot\vec{E}=0$)

b. (5 pts) What are the components of **E** and **B** for the lowest mode? Hint: the lowest mode has **E**-field in the z-direction only. FOR LOWEST MODE $E_X=E_Y=0$, $E_Z=E$. SINCE $\vec{\nabla}\times\vec{E}=-\partial\vec{B}/\partial t \rightarrow \vec{B}=-i/\omega\ \vec{\nabla}\times\vec{E}$; HENCE
$B_X=-i/\omega\ \partial E_Z/\partial Y,\quad B_Y=i/\omega\ \partial E_Z/\partial X,\quad B_X=0$

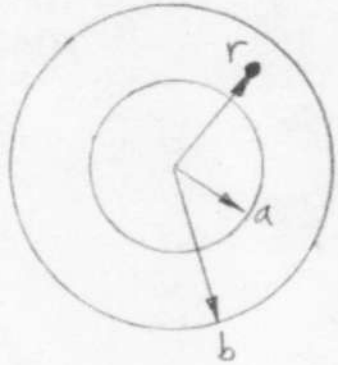
c. (5 pts) For this lowest mode, find the cutoff frequency.
$$f_{01}=\frac{c}{2}\sqrt{\left(\frac{1}{2\,\mathrm{cm}}\right)^2+\left(\frac{0}{1\,\mathrm{cm}}\right)^2}=7.5\ \mathrm{GHz}$$

d. (5 pts) If the volume inside the guide were filled with lossless plastic of relative dielectric constant $\varepsilon_r = 2$, how would the cutoff frequency change? THE VELOCITY IS SMALLER BY $\sqrt{\varepsilon_r}$, SO THE CUTOFF FREQUENCY DECREASES BY $\sqrt{\varepsilon_r}$. HENCE
$$f_{01}'=7.5\ \mathrm{GHz}/\sqrt{2}=5.4\ \mathrm{GHz}$$

e. (5 pts) The possible modes of propagation down the guide separate naturally into two classes. What are these two classes and describe how they differ. ONE CLASS HAS $\vec{E}$ TRANSVERSE, BUT $\vec{B}$ INCLUDES NON-ZERO B_X (TE). THE OTHER CLASS HAS $\vec{B}$ TRANSVERSE, BUT $\vec{E}$ INCLUDES NON-ZERO E_X (TM).

Name ______________________________ **Student ID** ____________ **Score**______
last *first*

II. Coaxial waveguide. A coaxial waveguide consists of cylindrical inner and outer perfect conductors of diameter *a* and *b* (see figure). The volume between conductors is vacuum. A TEM wave of angular frequency ω propagates down the guide. The amplitude of the voltage across the conductors is V_0, the amplitude of the current down one of the conductors is I_0.



a. (5 pts) What's the TEM cutoff frequency of the guide? Explain.

THE "CUTOFF" GOES DOWN TO DC; TEM MODES CAN SUPPORT A VOLTAGE DIFFERENCE ACROSS CONDUCTORS.

b. (5 pts) For a field point shown at radius r, find **E** in terms of V_0 and **B** in terms of I_0.

FOR FIXED TIME & POSITION ALONG LINE, WE HAVE 2D ELECTROSTATICS:

$$\vec{E} = V_0 / r \ln b/a \ \hat{r} \qquad \vec{B} = \mu_0 I_0 / 2\pi r \ \hat{\phi}$$

c. (5 pts) Find the "characteristic impedance" (V_0/I_0) of the guide. Hint: recall from homework the "wave impedance" (E_t/H_t) of the guide is $\sqrt{\mu_0/\varepsilon_0}$.

$$Z_0 = V_0/I_0 = \frac{E \ln b/a}{2\pi B/\mu_0} = \sqrt{\frac{\mu_0}{\varepsilon_0}} \frac{\ln b/a}{2\pi}$$

d. (5 pts) What value of resistor should you place across inner and outer conductors at the end of the guide to ensure there's no reflection from the end?

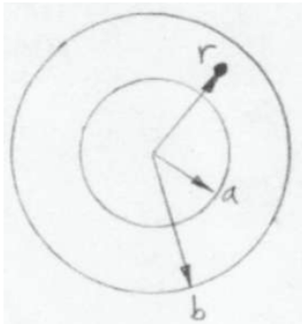
THE RESISTOR HAS VALUE $R = Z_0$

e. (5 pts) What's the time-average power delivered by the guide?

WE COULD EVALUATE THE TIME AVERAGE $\vec{E}$ & $\vec{B}$ FIELDS $\langle P \rangle = \frac{1}{2\mu_0} \int \vec{E} \times \vec{B}^* \cdot d\vec{s}$ OVER THE CROSS-SECTION. OR WE COULD RECALL

$$\langle P \rangle = \frac{1}{2} V_0 I_0$$

2. (25 pts total) *Coaxial Waveguide*
A coaxial waveguide consists of cylindrical inner and outer perfect conductors of diameters a and b (see figure). The volume between conductors is vacuum. A TEM wave of angular frequency ω propagates down the guide. The amplitude of the voltage across the conductors is V_0, the amplitude of the curreni down one of the conductors is I_0.



(a) (7 pts) What is the TEM cutoff frequency of the guide? Explain.
(b) (6 pts) For a field point shown at radius r, find $\mathbf{E}$ in terms of V_0 and $\mathbf{B}$ in terms of I_0.
(c) (6 pts) Find the "characteristic impedance" (V_0/I_0) of the guide.
(d) (6 pts) What is the time-averaged power delivered by the guide?

3. *Dispersive Gaseous Medium*
A dilute gaseous medium is found to exhibit a single optical resonance at frequency $\omega_0 = 2\pi\times10^{14}$ Hz. The electric field of a plane wave at frequency ω_0 propagating through this medium is attenuated by a factor of two over a distance of 10 meters. The frequency width of the absorption resonance is $\Delta\omega$.
(a) What is the absorption coefficient α on resonance?
(b) Arrange in ascending order the propagation velocities at frequencies ω_0, $\omega_0 + \Delta\omega/10$, and $\omega_0 - \Delta\omega/10$. Show your reasoning.
(c) If there were no other resonances in the medium, what are the approximate numerical values of the index of refraction and the propagation velocity on resonance?

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Solution to Sample Probs
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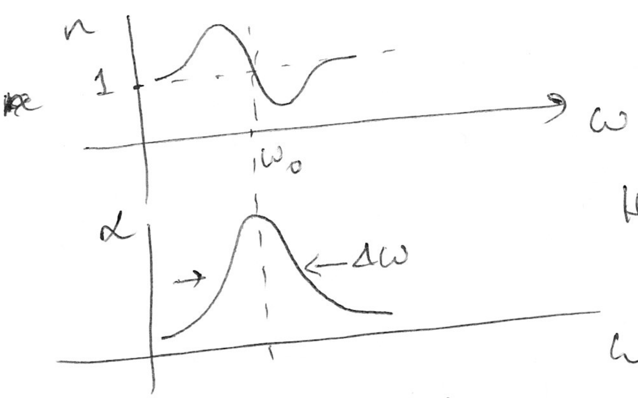
3]. Dispersive Gaseous Medium

$\omega_0 = 2\pi \times 10^{14}$ Hz.

For dilute gas, index of refraction (near resonance) $n \simeq 1 + \frac{1}{2\epsilon_0} \frac{Nfe^2}{m} \frac{(\omega_0^2 - \omega^2)}{(\omega_0^2-\omega^2)^2 + \gamma^2\omega^2}$

Absorption coefficient $\alpha \simeq \frac{\omega}{c\epsilon_0} \frac{Nfe^2}{m} \frac{\gamma\omega}{(\omega_0^2-\omega^2)^2+\gamma^2\omega^2}$

which look like:



Here $\gamma \sim \Delta\omega$.

(a). electric field is attenuated by a factor of 2 over 10 meters.

Since $E \propto e^{-Kz}$ electric field amplitude

& Intensity $I \propto E^2 \propto e^{-2Kz}$

we have: $e^{-K(10m)} = \frac{1}{2}$ on resonance

and $\alpha = 2K$. then $10K = \ln 2 \Rightarrow K = \frac{\ln 2}{10}$

and on resonance $\alpha = \frac{2\ln 2}{10} = \frac{\ln 2}{5} \simeq 0.14\ m^{-1}$

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3] (b). The propagation velocity or phase velocity is determined as

$$v_p = \frac{\omega}{k} = \frac{c}{n}.$$

Close to resonance, we have anomalous dispersion & $\frac{dn}{d\omega} < 0$ (see figure as well as eqn).

$$\Rightarrow \quad n\left(\omega_0 - \frac{\Delta\omega}{10}\right) > n(\omega_0) > n\left(\omega_0 + \frac{\Delta\omega}{10}\right)$$

$$\Rightarrow \quad v_p\left(\omega_0 - \frac{\Delta\omega}{10}\right) < v_p(\omega_0) < v_p\left(\omega_0 + \frac{\Delta\omega}{10}\right).$$

(c). No other resonances $\Rightarrow$ the equation for n can be used without considering the background contributions from other resonances.

Thus $\quad n(\omega_0) = 1 + 0 = 1$

$$v_p(\omega_0) = \frac{c}{n} = c \doteq 3 \times 10^8 \text{ m/s}.$$

These are approximate because the dilute gas limit expression comes from a Taylor expansion to 1st order (see Griffiths Eqn. 9.169 & surrounding text)

X