I. Rectangular waveguide. Consider the rectangular waveguide as shown. The guided axis is along the x-direction, with height (z-direction) 1 cm and width (y-direction) 2 cm. The volume inside is vacuum and the walls are perfect conductors. A wave propagates down the guide at angular frequency $\omega$.

![Diagram of a rectangular waveguide with dimensions labeled.](image)

a. (5 pts) Write the boundary conditions, if any, on $E_x$, $E_y$, $E_z$, $B_x$, $B_y$, and $B_z$ at the walls.

\[ E \text{ and } B \text{ vanish at surface. Hence:} \]
\[ (y = 0 \text{ or } 2 \text{ cm}) \quad B_y = 0, \quad E_x = E_z = 0, \quad \partial E_y / \partial y = 0 \quad (\text{from } \nabla \cdot E = 0) \]
\[ (z = 0 \text{ or } 1 \text{ cm}) \quad B_z = 0, \quad E_x = E_y = 0, \quad \partial E_z / \partial z = 0 \quad (\text{from } \nabla \cdot E = 0) \]

b. (5 pts) What are the components of $E$ and $B$ for the lowest mode? Hint: the lowest mode has $E$-field in the $z$-direction only.

For lowest mode $E_x = E_y = 0$,

$E_z = E$. Since $\nabla \times E = -\partial B / \partial t \Rightarrow \vec{B} = -i/\omega \vec{\nabla} \times E$; Hence

$B_x = -i/\omega \partial E_z / \partial y$, $B_y = i/\omega \partial E_z / \partial x$, $B_z = 0$

c. (5 pts) For this lowest mode, find the cutoff frequency.

\[ f_{01} = \frac{c}{2} \sqrt{\left(\frac{1}{2 \text{ cm}}\right)^2 + (0 \text{ cm})^2} = 7.5 \text{ GHz} \]

d. (5 pts) If the volume inside the guide were filled with lossless plastic of relative dielectric constant $\varepsilon_r = 2$, how would the cutoff frequency change? The velocity is smaller by $\sqrt{\varepsilon_r}$, so the cutoff frequency decreases by $\sqrt{\varepsilon_r}$. Hence

\[ f_{01}' = \frac{7.5 \text{ GHz}}{\sqrt{2}} = 5.4 \text{ GHz} \]

e. (5 pts) The possible modes of propagation down the guide separate naturally into two classes. What are these two classes and describe how they differ.

One class has $E$ transverse, but $B$ includes non-zero $B_x$ (TE). The other class has $B$ transverse, but $E$ includes non-zero $E_x$ (TM).
II. Coaxial waveguide. A coaxial waveguide consists of cylindrical inner and outer perfect conductors of diameter \( a \) and \( b \) (see figure). The volume between conductors is vacuum. A TEM wave of angular frequency \( \omega \) propagates down the guide. The amplitude of the voltage across the conductors is \( V_0 \), the amplitude of the current down one of the conductors is \( I_0 \).

![Coaxial waveguide diagram]

a. (5 pts) What's the TEM cutoff frequency of the guide? Explain.

The "cutoff" goes down to DC; TEM modes can support a voltage difference across conductors.

b. (5 pts) For a field point shown at radius \( r \), find \( E \) in terms of \( V_0 \) and \( B \) in terms of \( I_0 \).

For fixed time & position along guide, we have 2D electrostatics:

\[
E = \frac{V_0}{r} \ln \frac{b}{a} \hat{r} \quad B = \frac{I_0}{2\pi r} \hat{\phi}
\]

c. (5 pts) Find the "characteristic impedance" \( (V_0/I_0) \) of the guide. Hint: recall from homework the "wave impedance" \( (E_0/H_0) \) of the guide is \( \sqrt{\mu_0/\varepsilon_0} \).

\[
Z_0 = \frac{V_0}{I_0} = \frac{E \ln \frac{b}{a}}{2\pi B / I_0} = \frac{\sqrt{\mu_0}}{\varepsilon_0} \frac{\ln \frac{b}{a}}{2\pi}
\]

d. (5 pts) What value of resistor should you place across inner and outer conductors at the end of the guide to ensure there's no reflection from the end?

The resistor has value \( R = Z_0 \)

e. (5 pts) What's the time-average power delivered by the guide?

We could evaluate the time average \( \overline{E^2 B^2} \) fields \( \langle P \rangle = \frac{1}{2\mu_0} \int \overline{E \times B^*} \cdot \overline{dB} \) over the cross-section. Or we could recall

\[
\langle P \rangle = \frac{1}{2} V_0 I_0
\]
2. (25 pts total) *Coaxial Waveguide*

A coaxial waveguide consists of cylindrical inner and outer perfect conductors of diameters $a$ and $b$ (see figure). The volume between conductors is vacuum. A TEM wave of angular frequency $\omega$ propagates down the guide. The amplitude of the voltage across the conductors is $V_0$, the amplitude of the current down one of the conductors is $I_0$.

(a) (7 pts) What is the TEM cutoff frequency of the guide? Explain.
(b) (6 pts) For a field point shown at radius $r$, find $E$ in terms of $V_0$ and $B$ in terms of $I_0$.
(c) (6 pts) Find the “characteristic impedance” ($V_0/I_0$) of the guide.
(d) (6 pts) What is the time-averaged power delivered by the guide?

3. *Dispersive Gaseous Medium*

A dilute gaseous medium is found to exhibit a single optical resonance at frequency $\omega_0 = 2\pi \times 10^{14}$ Hz. The electric field of a plane wave at frequency $\omega_0$ propagating through this medium is attenuated by a factor of two over a distance of 10 meters. The frequency width of the absorption resonance is $\Delta\omega$.

(a) What is the absorption coefficient $\alpha$ on resonance?
(b) Arrange in ascending order the propagation velocities at frequencies $\omega_0$, $\omega_0 + \Delta\omega/10$, and $\omega_0 - \Delta\omega/10$. Show your reasoning.
(c) If there were no other resonances in the medium, what are the approximate numerical values of the index of refraction and the propagation velocity on resonance?
3) Dispersive Gaseous Medium

\[ \omega_0 = 2\pi \times 10^{14} \text{ Hz} . \]

For dilute gas, index of refraction \( n \approx 1 + \frac{1}{2\varepsilon_0 m} \frac{N\varepsilon_0^2}{\omega_0^2} \frac{\omega^2 - \omega_0^2}{(\omega^2 - \omega_0^2)^2 + \delta^2 \omega^2} \) (near resonance)

Absorption coefficient \( \alpha = \frac{\omega N\varepsilon_0^2}{\delta\varepsilon_0 m} \frac{\delta\omega}{(\omega^2 - \omega_0^2)^2 + \delta^2 \omega^2} \)

which look like:

\[ n \begin{array}{c}
\Re \\
1 \\
\omega_0 \\
\alpha \\
\omega
\end{array} \]

Here \( \delta \sim \Delta \omega \).

(a) Electric field is attenuated by a factor of 2 over 10 meters.

Since \( E \propto e^{-Kz} \) electric field amplitude & intensity \( I \propto E^2 \propto e^{-2Kz} \)

we have: \( e^{-K(10 \text{ m})} = \frac{1}{2} \) on resonance

and \( K = 2K \). Then \( 10K^2 = \ln 2 \Rightarrow K = \frac{\ln 2}{10} \)

and on resonance \( \alpha = 2\ln 2 = \frac{\ln 2}{5} \approx 0.14 \text{ m}^{-1} \)
3] (b). The propagation velocity or phase velocity is determined as
\[ V_p = \frac{\omega}{k} = \frac{c}{n}. \]

Close to resonance, we have anomalous dispersion and
\[ \frac{dn}{d\omega} \approx 0 \quad (\text{see figure as well as Eqs.}). \]
\[ \Rightarrow \quad n(\omega - \Delta \omega) > n(\omega) > n(\omega + \Delta \omega). \]
\[ \Rightarrow \quad V_p(\omega - \Delta \omega) < V_p(\omega) < V_p(\omega + \Delta \omega). \]

(c) No other resonances \( \Rightarrow \) the equation for \( n \) contributions can be used without considering the background from other resonances.

Thus, \( n(\omega) = 1 + 0 = 1 \)
\[ V_p(\omega) = \frac{c}{n} = c = 3 \times 10^8 \text{ m/s}. \]

There are approximate because the dilute gas limit expression comes from a Taylor expansion to 1st order (see Griffiths Eqs. 9.169 & surrounding text).