PHYSICS 323:
ELECTROMAGNETISM

21 April 2020    Midterm 1 Solution


A plasma is an ionized gas consisting at least partly of free electrons (of mass $m$) and positively charged ions; it therefore can be a conducting material. There are no bound dipole moments, and the medium is non-magnetic ($\epsilon = \epsilon_0, \mu = \mu_0$). The sun and stars are largely plasmas. Consider the classical model discussed in class and in Sect. 9.4.3, with $\gamma = 1/\tau$, and the electrons are not bound ($\omega_0 = 0$). You may ignore the effects of any positively charged ions.

(a) (10) The density of the electrons in the plasma is $N$ and the current density $J = -eNv = \sigma \mathbf{E}$. Take the electric field to have the form $\mathbf{E}(t) = E_0 e^{-i\omega t}$. Show that the conductivity is given by $\sigma(\omega) = \frac{i\epsilon_0 \Omega^2}{\omega + i\gamma}$ with $\Omega^2 \equiv \frac{Ne^2}{m\epsilon_0}$.

Use $m \mathbf{a} = \mathbf{F}$, so $m \ddot{\mathbf{r}} + m\gamma \dot{\mathbf{r}} = -e\mathbf{E}_0 e^{-i\omega t}$. Assume $\mathbf{r} = \mathbf{r}_0 e^{-i\omega t}$ to get $(-m\omega^2 - im\gamma)\mathbf{r}_0 = \mathbf{E}_0$, $\mathbf{v} = \dot{\mathbf{r}} = -i\omega \mathbf{r}_0 = \frac{iwm\mathbf{E}_0}{(m\omega^2 + im\gamma)} = -i\frac{e\mathbf{E}_0}{m(\omega + i\gamma)}$. There are $N$ electrons per unit volume so $\sigma(\omega) = -eN(-ie/m)/(\omega + i\gamma) = \frac{i\Omega^2 \epsilon_0}{\omega + i\gamma}$.

(b) (5) Now consider a plane wave with the form $\mathbf{E}(t) = \mathbf{E}_0 e^{i(kz - \omega t)}$ traveling through a medium with the stated $\sigma(\omega)$. Compute the value of $k$ (appropriate for propagation in the positive $z$ direction) in terms of the given quantities. You may express your answer in terms of a well-defined square root.

The wave equation (9.122) is $\nabla^2 \mathbf{E} = \frac{1}{c^2} \frac{\partial^2 \mathbf{E}}{\partial t^2} + \mu_0 \sigma(\omega) \frac{\partial \mathbf{E}}{\partial t}$, using the given plane wave form yields $-k^2 = -\frac{\omega^2}{c^2} - i\omega\mu_0 \sigma(\omega)$ so $k = \sqrt{\frac{\omega}{c} \sqrt{1 + \frac{i\sigma(\omega)}{\epsilon_0 \omega}}}$.

(c) (5) For a dilute plasma the damping factor $\gamma$ is very small, so take $\gamma = 0.01\Omega$, and for a typical plasma $\Omega = 1.5 \times 10^{10}$ rad/s. For what frequencies, $\omega$, is $\text{Re}(\sigma) \gg \text{Im}(\sigma)$. $\sigma = \frac{\Omega^2 \epsilon_0}{\omega^2 + \gamma^2} i(\omega - i\gamma) = +\frac{\Omega^2 \epsilon_0}{\omega^2 + \gamma^2} (i\omega + \gamma)$. The real part is dominant if $\omega \ll \gamma = 1.5 \times 10^8$ rad/s.

Extra -not asked for The wave is damped if the real part of $\sigma$ is much larger than the imaginary part. This means that the wave is damped for low values of $\omega$. 

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2. *Total reflection* Consider the case that light propagates from a dense optical medium, \( n > 1 \), into air with \( n = 1 \). See Fig. 9.14, with the left-hand side of \( n > 1 \) and the right-hand side with air, \( n = 1 \). Both sides have \( \mu = \mu_0 \).

(a) (5) For what angles \( \theta_I \) is the angle of transmission (angle of refraction) \( \theta_T \) a complex number? In the present case Snell’s law says \( n \sin \theta_I = \sin \theta_T \). If \( n \sin \theta_I > 1 \), \( \sin \theta_T > 1 \) and \( \theta_T \) must be a complex number.

(b) (10) Consider the wave that moves in the air for situations in which \( \theta_T \) is a complex number. Show that the wave propagates along the \( x \)-direction, but falls off exponentially in the \( z \) direction.

The wave moving through air behaves as \( e^{i(k \cdot r - \omega t)} \). Examine the first term of the exponent: \( ik \cdot r = ik(x \sin \theta_T + z \cos \theta_T) \). The quantity \( \sin \theta_T \) is real because both \( n \) and \( \sin \theta_I \) are real numbers. Thus there is propagation in the \( x \) direction. The quantity \( \cos \theta_T = \sqrt{1 - \sin^2 \theta_T} \). But \( \sin \theta_T > 1 \) part (a). So \( \cos \theta_T \) is the square root of a negative number and is purely imaginary. Thus the factor in the exponent multiplying \( z \) is a negative real number, so the wave is damped in the \( z \) direction.

(c) (5) For the situations of part (b), compute the ratio of the intensity of the reflected to that of the incident wave for the case of incident waves polarized parallel to the plane of incidence.

Use 9.107 \( \frac{E_{0R}}{E_{0I}} = \left( \frac{\alpha - \beta}{\alpha + \beta} \right) \). \( \alpha = \cos \theta_T / \cos \theta_I \), \( \beta = 1/n = \sin \theta_I / \sin \theta_T \), so

\[
\frac{E_{0R}}{E_{0I}} = \left( \frac{\cos \theta_T \sin \theta_T - \sin \theta_I \cos \theta_T}{\cos \theta_T \sin \theta_T - \sin \theta_I \cos \theta_T} \right).
\]

The quantities \( \sin \theta_I, \cos \theta_I, \sin \theta_T \) are real numbers, but \( \cos \theta_T = \sqrt{1 - \sin^2 \theta_T} \) is purely imaginary because \( \sin \theta_T > 1 \). Thus the ratio \( \frac{E_{0R}}{E_{0I}} \) is of the form \( \frac{\alpha - \beta}{\alpha + \beta} \) where \( X, Y \) are real numbers. The ratio of the reflected to incident intensities is the absolute square: \( |\frac{E_{0R}}{E_{0I}}|^2 = \frac{Y^2 + X^2}{Y^2 + X^2} = 1 \).
3. **Cylindrical wave guide** Consider a hollow cylindrical wave guide of radius $a$. The cylinder is made of a perfect conductor. The aim is to learn about the TM modes that propagate down the cylinder in the $z$ direction. Use cylindrical coordinates and write $E_z(r, t) = \psi(s, \phi)e^{i(kz-\omega t)}$.

(a) (5) What is $\psi(a, \phi)$?

The reason (NOT required for full credit) is that the tangential components of the electric field vanish along the surface of perfect conductor. This surface is an equipotential.

(b) (8) Explain why a solution $\psi(s, \phi)$ can be written as $R(s)e^{im\phi}$ with $m$ an integer (1), and obtain a differential equation for $R(s)$ (7). The value of the electric field must not change when $\phi$ is changed by the integer number of $2\pi$, and $e^{im\phi} = e^{im(\phi+2n\pi)}$, when $m, n$ are integers.

Use 9.181 in cylindrical coordinates (inside front cover)

\[
\left( \frac{1}{s} \frac{\partial}{\partial s} s \frac{\partial}{\partial s} + \frac{1}{s^2} \frac{\partial^2}{\partial \phi^2} + \frac{\omega^2}{c^2} - k^2 \right) \psi(s, \phi) = 0 \quad \text{Use } \psi(s, \phi) = R(s)e^{im\phi} \text{ and } \gamma^2 \equiv \frac{\omega^2}{c^2} - k^2. \quad \text{Then }
\]

\[
\left( \frac{d^2}{ds^2} + \frac{1}{s} \frac{d}{ds} - \frac{m^2}{s^2} + \gamma^2 \right) R(s) = 0,
\]

getting to the above is worth full credit. The following can be in part(b) or part(c)

Divide the above equation by $\gamma^2$ and define $x = \gamma s$. Then find $(\frac{d^2}{dx^2} + \frac{1}{x} \frac{d}{dx} - \frac{m^2}{x^2} + 1)R(s) = 0$, that is Bessel’s differential equation

(c) (7) Find the lowest cutoff frequency.

The regular solutions of the equation $\frac{d^2R}{dx^2} + \frac{1}{x} \frac{dR}{dx} + (1 - \frac{m^2}{x^2})R = 0$ are denoted Bessel functions $J_m(x)$.

The regular solutions of the differential equation written above in part (b) are $J_m(\gamma a) = J_m(\gamma s)$. We must have $J_m(\gamma a) = 0$ by the boundary condition of part (a). The lowest cutoff frequency is defined by the lowest value of $\omega$ for which $k^2 = \frac{\omega^2}{c^2} - \gamma^2 > 0$. This is determined by the lowest value of $\gamma$. Looking up the zeros of Bessel functions, the smallest value of $\gamma a = 2.405$ that occurs for $m = 0$. Thus the lowest cutoff frequency is $\boxed{2.405/a}$. 
