There are four problems in this exam, each with several parts. Each problem is worth 20 points.

This exam is open book, open notes, open internet, open calculator, and open computer, but the work must be your very own. Therefore you must sign your name below.

I certify that the submitted answers are my own work, done without collaboration

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Sign your name on the line above.

Read the problems carefully.
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1. **Potentials and Gauges**

(a) (7) Suppose that you are given the vector potential in the Lorentz gauge at one point in space, for all relevant times as \( \mathbf{A} = A_0 \cos \omega t \), where \( A_0 \) is a constant vector. Could you determine the scalar potential at that same point? Explain why or why not.

(b) (7) Suppose instead that the Lorentz-gauge vector potential in a finite region of space is given by \( \mathbf{A}(\mathbf{r}, t) = A_0 \frac{r}{R} \cos \omega t \), where \( A_0 \) is a constant with units of the vector potential and \( R \) is a given length. Determine the scalar potential (for 5 points) and the charge density (for 2 points) in the same finite region of space.

(c) (6) Suppose you are given potentials \( V_C(\mathbf{r}, t) \) and \( \mathbf{A}_C(\mathbf{r}, t) \) that are expressed in the Coulomb gauge. Derive a well-defined equation for the potentials \( V_L, A_L \) in the Lorentz gauge. The answer should be in the form of a three-dimensional integral.
2. Vector Potential in the Radiation Zone

A system has a current density: \( \mathbf{J}(\mathbf{r}, t) = \frac{J_0}{R} (-y \hat{i} + x \hat{j}) \exp \left(-\frac{r^2}{R^2}\right) \cos \omega t \), where \( \hat{i} \) is the unit vector in the \( x \)-direction, \( \hat{j} \) is the unit vector in the \( y \)-direction, \( J_0 \) is a given constant, and \( R \) is a given length.

(a) (5) Determine the time rate of change of the charge density.

(b) (10) Assume that the long wave-length limit, \( \omega R/c \ll 1 \) holds. Compute the vector potential \( \mathbf{A}(\mathbf{r}, t) \) caused by the given current density at positions \( \mathbf{r} \) in the radiation zone.

Hint: \( \int_{-\infty}^{\infty} dz \exp \left(-\frac{z^2}{R^2}\right) = \sqrt{\pi} R \), \( \int_{-\infty}^{\infty} dz z^2 \exp \left(-\frac{z^2}{R^2}\right) = \frac{\sqrt{\pi} R^3}{2} \)

(c) (5) Determine the magnetic field for positions in the radiation zone along the \( z \) axis. Explain.
3. Angular distribution of radiated power for two antennas

Consider two systems of the type in Sect. 11.1.2, except here the long wave length approximation is NOT applicable. This means that using Eq. (11.17) will not lead to the correct answer. One system is at \((x, y, z) = (L/2, 0, 0) \equiv r_1\) and the other at \((x, y, z) = (-L/2, 0, 0) \equiv r_2\). The charge oscillates up and down the \(z\) axis according to Eq. (11.3) for both elements. The length \(L\) is of the same order of magnitude as the height of the system, \(d\).

(a) (10) Determine the vector potential in the radiation zone: \(r \gg L, r \gg d\).

(b) (5) Determine the magnetic field in the radiation zone.

(c) (5) For a fixed value of \(r \gg L, r \gg d\) find all positions where the angular distribution of radiated power vanishes.
IV. [20 points total] Tutorial question.

A positive point charge oscillates sinusoidally along the z-axis between the origin and 
z = 1 lightsecond (ls) with a period of 8 seconds, such that the charge is at the origin when 
t = {0s,8s,16s,...} and at z = 1 ls when t = {4s,12s,20s,...}.

A. Consider an observer on the z-axis at z = 5 ls.
   i. [5 pts] At the instant t = 16s, at what retarded time would the observer describe the point 
      charge? If the exact retarded time cannot be determined precisely, you may give a range 
      between two consecutive integers, non-inclusive (e.g., tret between 1s and 2s). Explain your reasoning.

   ii. [5 pts] Compare the scalar potential in the Coulomb Gauge, \( V_{\text{Coulomb}} \), to the scalar potential in 
       the Lorentz Gauge, \( V_{\text{Lorentz}} \). At z = 5 ls and t = 16s (the same location and time as above), is 
       \( V_{\text{Coulomb}} \) greater than, less than, or equal to \( V_{\text{Lorentz}} \)? Explain your reasoning.

B. [10 pts] Consider points very far from the origin, first along the x-axis and then along the z-axis.

   Once the time-averaged electric field is expanded in terms of \( \left( \frac{1}{x} \right)^n \) and \( \left( \frac{1}{z} \right)^n \), respectively, what is 
   the power of the first non-zero term in each expansion? (I.e., what is the first \( n_x \) for the x-axis and 
   what is the first \( n_z \) for the z-axis?) Explain your reasoning.