Consider propagation in region with no free charges or currents:

$\nabla \cdot \mathbf{B} = 0 \quad \nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$

$\nabla \cdot \mathbf{H} = 0 \quad \nabla \times \mathbf{H} = -\frac{\partial \mathbf{D}}{\partial t}$

Try to solve can't.

Need relation between $\mathbf{B}, \mathbf{H}, \mathbf{E}, \mathbf{D}$.

Linear medium:

$\mathbf{D} = \varepsilon \mathbf{E}$

$\mathbf{B} = \mu \mathbf{H}$

Then $\nabla \mathbf{D} = 0 \quad \nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$

$\nabla \cdot \mathbf{H} = 0 \quad \nabla \times \mathbf{H} = -\frac{\partial \mathbf{D}}{\partial t}$

The same as before $\varepsilon \mu_0 \rightarrow \varepsilon \mu$.

$C^2 = \frac{1}{\varepsilon \mu_0}$ now waves propagate.

with $\nu = \frac{1}{\sqrt{\mu \varepsilon}} = \frac{c}{\mu} \quad c = 1 \mu_0 \varepsilon_0$

of refraction $n = \sqrt{\frac{\mu_0}{\varepsilon_0}} \times \sqrt{\frac{\varepsilon}{\mu}} = \sqrt{\varepsilon_0}$.
All previous formula carry over with the replacement $\varepsilon_0 \rightarrow \varepsilon$, $\mu_0 \rightarrow \mu$

Energy density

$$U = \frac{1}{2} \left( \mathbf{E} \cdot \mathbf{E} + \mathbf{B} \cdot \mathbf{B} \right) \frac{1}{\mu}$$

$$\mathbf{S} = \frac{1}{\mu} \left( \mathbf{E} \times \mathbf{B} \right) = \left( \mathbf{E} \times \mathbf{H} \right)$$

$$I = \frac{1}{2} \varepsilon_0 E_0^2$$

---

Turn to another topic.

So far, we've been propagating in a uniform medium. Let's now consider what happens when light hits a boundary between two media—light passing from air to water or wine. First, consider an easy case: normal incidence.
What happen Suppose line boundary

\[ E_1, \mu_1 = \rho_0 \]
\[ \nu_1 = \frac{c}{\nu_1} \]
\[ E_2, \mu_2 = \rho_0 \]
\[ \nu_2 = \frac{c}{\nu_2} \]

Light comes in from left. What happen when hit boundary?

Recall string - part goes thru, part bounce back. How much need to calculate.

We'll do a special case

First normal incidence

Suppose light incident from left. Pol in x direction.

\[ E_x (2t) = \frac{E_o}{\nu_1} e^{i(k_1 x - \omega t)} \]

\[ B_y (2t) = \frac{E_o}{\nu_1} e^{i(k_1 x - \omega t)} \]

What if
There will be reflected wave
\[ E_R(z,t) = E_{or} e^{i(-kz - \omega t)} \]
and transmitted wave
\[ E_T(z,t) = E_{ot} e^{i(kz - \omega t)} \]

Goal: \( E_{or} \), \( E_{ot} \)
are the only unknowns.

How do we do that?

Boundary conditions
Have no free charge or current at boundary everywhere.
Tangential components \( \mathbf{E} \), \( \mathbf{D} \) normal to \( \mathbf{B} \).

How apply this idea
Note: in medium \( \mathbf{D} \) total \( \mathbf{E} \)
Entire sum of incident + reflected waves.
\( I_1 = \frac{1}{2} \varepsilon_1 \pi \frac{E_0^2}{\varepsilon_1^2} \)

**Reflection coefficient**

\[
R = \frac{I_R}{I_I} = \left| \frac{E_{op}}{E_{in}} \right|^2 = \left( \frac{n_1 - n_2}{n_1 + n_2} \right)^2
\]

\[
T = \frac{I_T}{I_I} = T = \text{transmission coefficient}
\]

We expect \( R + T = 1 \) no thing is lost here no light

Then \( T = 1 - \frac{(n_1 - n_2)^2}{\varepsilon_1 (\varepsilon_1 n_2)} \)

\[
= \frac{\left( n_1 (\varepsilon_1 n_2)^2 - (n_1 - n_2)^2 \right)}{(n_1 \varepsilon_1 n_2)} = \frac{4n_1 n_2}{\varepsilon_1 (\varepsilon_1 n_2)}
\]

Sanity check \( n_1 = n_2 \)

Check with direct calculation

\[
T = \frac{\varepsilon_2 \varepsilon_1}{\varepsilon_1 n_1} \left| \frac{E_{op}}{E_{in}} \right|^2 = \frac{n_2}{n_1} \left( \frac{4n_1 n_2}{\varepsilon_1 \varepsilon_2} \right)^2 \]

\[
\left( \frac{n_2}{n_1} \right)^2 ( )
\]

So agrees
\[ E \text{ is entirely tangential so is } \vec{B} = \mu_0 \vec{H} \]

\[ z = 0 \]

\[ \vec{E}_{OI} + \vec{E}_{OR} = \vec{E}_T \]

\[ \frac{1}{2} (\vec{E}_{OZ} - \vec{E}_{OR}) = \frac{1}{2} \vec{E}_T \]

two eqns 2 unknowns

\[ \beta = \frac{V_1}{V_2} \]

\[ 2 \vec{E}_{OZ} = \vec{E}_T (1 + \beta) \]

\[ \vec{E}_T = \frac{2}{1 + \beta} \vec{E}_{OZ} \]

\[ \vec{E}_{OR} = \vec{E}_T - \vec{E}_{OR} = \frac{2V_2 - \beta V_1}{V_1 + V_2} \vec{E}_{OZ} \]

\[ \vec{E}_{OR} = \frac{1 - \beta}{1 + \beta} \vec{E}_{OZ} \]

How to check: \( \lim_{\beta \to 1} \)

OK

\[ V_2 > V_1 \quad R \text{ is in phase} \]

\[ V_1 > V_2 \quad \text{Phase change} \]

\[ \text{low freq. to high freq.} \]
Now we do the general case of oblique angle

![Diagram of an incident wave and a reflected wave]

**Incident wave**

**Reflected wave**

**What question?**

**Given** \( \theta_i \), **want** \( \theta_e, \theta_r \)

Also will want direction of \( \vec{E}_2, \vec{H}_r \)

This gives polarization of light

---

Will use boundary conditions:

tangential \( \vec{E} \), \( \vec{H} \) continue

normal \( D, B \) continue

First step: Enumerate fields

**Incident wave**

\[
\vec{E}_i = \vec{E}_0 e^{i(kz - \omega t)}
\]

\[
\vec{B}_i = \frac{-\vec{E}_0}{\omega} e^{i(kz - \omega t)}
\]

**Reflected wave**

\[
\vec{E}_r = \vec{E}_0 e^{i(kz - \omega t)}
\]

\[
\vec{B}_r = \frac{\hat{r} \times \vec{E}_r}{\omega}
\]

\( \vec{r} \) not \( \perp \) to \( \hat{z} \)

Need to figure out where \( \vec{r} \) goes
Transmitted \( \hat{E}_T = \hat{E}_0 e^{-j(\omega t - \omega x)} \)

\[ \hat{B}_T = \frac{\hat{E}_T \times \hat{E}_0}{\hat{E}_0} \]

\[ -\frac{1}{c^2} \]

Notice the same \( e^{-j\omega t} \) is everywhere.
This is the only way to satisfy the boundary condition.

Advantage of complex notation: Get mag of \( k_{1,2} \).
So we may write \( \hat{h}_{1,2} = \hat{h}_p = \frac{\mu_0}{c} \sqrt{\epsilon_r} = \frac{\omega}{c} n_1 \)

\[ \hat{h}_T = \frac{\omega}{c} m_2 \]

Can get a lot of info from mere existence of boundary conditions BC.

The interface is at \( z = 0 \).

The BC must be satisfied at all values of \( x, y \) on the interface.

Why? If not true BC can't be satisfied.

This means at \( z = 0 \):

\[ \hat{E}_T \cdot \hat{r} = \hat{h}_p \cdot \hat{r} = \hat{h}_T \cdot \hat{r} \]

\[ = 0 \]

\( \hat{r} \) refers to position in complex plane.

Must be true at one place to satisfy BC. But that...
place can be anywhere

Example from Griffiths

If above is true at one point
changing $\ell$ will make it not true.

Another way to model problem

$$A e^{i\alpha x} + B e^{i\beta x} + C e^{i\gamma x} \quad (\text{like our problem})$$

show

and $a = b + c$

so

$$A (e^{i\alpha x} - e^{i\gamma x}) + B (e^{i\beta x} - e^{i\gamma x}) = 0$$

Take $\frac{d}{dx}$ at $x = 0$

$$A (a - c) = -B (b - c) \quad (1)$$

Take $\frac{d^2}{dx^2}$ at $x = 0$

$$A (a^2 - c^2) = -B (b^2 - c^2) \quad (2)$$

Take (2)

$$A (a + c) = (b + c)$$

$$\Rightarrow a = b$$

$$A (a - c) = -B (a - c)$$

either $c = a$ or $A = -B$

if $A = -B \quad c = 0 \quad \text{which violates}$

so $c = a = b$
So use BC idea to relate I or R

\[ x(h_1)_x + y(h_2)_y = x(h_1)_x + y(h_2)_y = x(h_1)_x + y(h_1)_y \]

Can set \( x = 0 \) \[ (h_2)_y = (h_2)_y = (h_1)_y \]

\[ y = 0 \] \[ (h_2)_x = h_2x = h_1x \]

Look at interface plane

Side:

Y axis out of page \( h_1 \) in \( x, z \) plane, \( h_2 \) in \( x, z \) plane, \( h_2y = 0 \)

Thus \( h_1y = 0 = h_2x = h_1y \)

Can work in \( x, z \) plane, plane of incidence

\( h_1, h_2 \) in same plane just as drawn

Look at \( x \) comp of \( f(h_2)_x = h_1 \sin \theta \)

\[ \theta = \theta \]

\[ \sin \alpha \] angles go from 0 to \( \pi/2 \)

Also \( \theta \) \[ h_1 \sin \theta = h_2 \sin \theta \]

\[ \theta \]

\[ \frac{h_2}{h_1} \sin \theta = \frac{h_2}{h_1} \sin \theta \]

\[ \sin \theta \]
$n_1 \sin \theta_1 = n_2 \sin \theta_2$

Snell's Law

These 2 results are true for any type of wave: sound, seismic, EM.

Also, marching band moving at angle to a boundary between low and high grass.

But EM is much richer. $E \times B$ can have

$x$, $y$ components

Let's be specific about BC. There are no free charges or free currents.

Tangential $E$ is continuous.\[ (E_{01} + E_{02})_{x,y} = (E_{01})_{x,y} \]

Normal $D$ is continuous.\[ E_1 [E_{01} + E_{02}]_z = E_2 (E_{01})_z \]

be cause we have no free charges on the dielectric surface.

Don't bother with $B$'s given $E$'s known B.
To proceed we need to specify polarization of light. There are two separate cases, depending on whether light is polarized or not to plane of incidence. Why different cases?

Physics is oscillating dipole at surface. \( \mathbf{P} \propto \mathbf{E} \)

Suppose \( \mathbf{E} \) incident light is polarized \( \parallel \) to plane of incidence. Means \( \mathbf{E} \) has only \( x \) component.

\[ E_0 \cos \theta_i + E_0 \cos \theta_l = E_0 \cos \theta_t \]

Two equations two unknowns.
Again, we have 2 equations unknown
Simplify equations whatever

\[ \beta = \frac{n_2}{n_1} \]

\[ d = \frac{\cos \Theta_i}{\cos \Theta_i} \]

Solve 2 equations unknowns

\[ E_{or} = \left( \frac{d - \beta}{d + \beta} \right) E_{oi} \]

\[ E_{oi} = \frac{2}{d + \beta} E_{oi} \]

Fresnel equations

Transmitted wave in phase with incident wave

Reflected wave is in phase if \( d > \beta \)
Or \( 180^\circ \) out of phase if \( d < \beta \)

\( d, \beta \) depend on indices of refraction

\( \beta \) (obvious!)

Let's look at \( d \)

\[ d = \frac{1 - \sin \Theta_i}{\cos \Theta_i} = \frac{\sqrt{1 - \left( \frac{n_1}{n_2} \right)^2 \sin^2 \Theta_i}}{\cos \Theta_i} \]
How can we check Equ:

\[
\text{normal incidence } \theta_1 = 0 \quad \Rightarrow \quad d = 1 \quad \theta_1 = 20 \times 2\pi
\]

get the previous result

\[
\tan \theta_2 = \frac{n_i}{n_i}
\]

grazing incidence \( \theta_2 = 90^\circ \)

\[
d \to 0 \quad \text{entire wave is reflected}
\]

For \( \theta_2 = 90^\circ \), \( E_r \to 0 \)

Can the reflected wave be eliminated

\( d = \beta \) means

\[
\cos \theta_2 = \frac{n_i}{n_i} = \sqrt{1 - \left( \frac{n_i}{n_i} \sin \theta_2 \right)^2} = \theta
\]

\[
\frac{n_i}{n_i} \quad \text{when this can satisfied } \theta_1 = \theta_2 \quad \text{- Brewster angle}
\]

\[
\frac{n_i^2}{n_i^2} \quad \text{or} \quad \cos^2 \theta_b = 1 - \frac{n_i^2 \sin^2 \theta_b}{n_i^2}
\]

\[
\tan \theta_2 =
\]

\[
\beta^2 \cos^2 \theta_b = 1 - \left( \frac{1}{\beta^2} \right) \sin^2 \theta_b
\]

\[
\beta^2 - \beta^2 \sin^2 \theta_b = 1 - \frac{n_i^2 \sin^2 \theta_b}{n_i^2}
\]

\[
\frac{1 + \beta^2}{\beta^2} = \sin^2 \theta_b = \frac{\beta^4 - \beta^2}{\beta^2 - 1} \quad \text{or} \quad \tan \theta_2 = \beta
\]

\[
\beta^2 - 1 < \theta - \frac{1}{\beta^2}
\]

\[
\beta \geq \frac{n_i}{n_i} \quad \beta > 1
\]
\[ \text{from } \text{air } n_1 = 1 \text{ to } \text{glass } n_2 = 1.5 \]

\[ \beta = 1.5 \]

\[ \theta_2 = 0 \]

\[ I_1 = \frac{1}{2} \epsilon_1 \epsilon_0 E_0^2 \cos \theta_2 \]

Power per unit area, strike boundary

\[ S \]

\[ R = \frac{I_R}{I_I} = \left| \frac{E_{02}}{E_{01}} \right|^2 = \left( \frac{k - \beta}{k + \beta} \right)^2 \]

\[ T = \text{more complicated} \]

\[ T = \frac{I_T}{I_I} = \frac{1}{2} \left( E_{02} \right)^2 \cos \theta_2 = \frac{1}{2} \beta \frac{4}{(k + \beta)^2} \]

\[ R + T = 1 \]

Won't do \( T \) case - nice extension for possible \( \theta \)
\[ E_T = E_{zo} \hat{e}_{y} e^{i (k_z \cdot r - \omega t)} \]
\[ B_z = \frac{\hat{e}_x \times E_T}{\mu_0} \]
\[ \hat{B}_z = \sin \theta \hat{e}_{\phi} + \cos \theta \hat{e}_{z} \]
\[ E_r = E_{ro} \hat{e}_r e^{i (k_r \cdot r - \omega t)} \]
\[ \hat{E}_r = E_{ro} \hat{e}_r e^{i (k_r \cdot r - \omega t)} \]

**BC**
- **Tangential**: Continuity
  \[ E_{zo} + E_{ro} = E_{zo} \]

**Normal**
- \( \vec{D} \) is continuous
  \[ \vec{B} = \vec{e}_r \vec{E} + \vec{e}_z \vec{D} \]
- \( \vec{B} \) has no flux \( \mu_0 \theta \)
- Normal component need another eq.

**BC**
- **Normal** \( \vec{B} \)