Propagation in matter

Consider propagation in matter region with no free charges or currents

$$\vec{\nabla}\cdot\vec{D} = 0 \qquad \vec{\nabla}\times\vec{E} = -\frac{\partial\vec{B}}{\partial t}$$

$$\vec{\nabla}\cdot\vec{B} = 0 \qquad \vec{\nabla}\times\vec{H} = -\frac{\partial\vec{D}}{\partial t}$$

Try to solve can't

Need relation between B, H, E, D

Linear medium

$$\vec{D} = \epsilon\vec{E}$$
$$\vec{B} = \mu\vec{H}$$

Then $\vec{\nabla}\cdot\vec{E} = 0 \qquad \vec{\nabla}\times\vec{E} = -\frac{\partial\vec{B}}{\partial t}$

$$\vec{\nabla}\cdot\vec{B} = 0 \qquad \vec{\nabla}\times\vec{B} = -\frac{\partial}{\partial t}\mu\epsilon\vec{E}$$

Carry out same derivation $\nabla^2\vec{E} = \mu\epsilon\frac{\partial^2\vec{E}}{\partial t^2}$

Same as in the space ~~here~~ $\mu_0\epsilon_0 \rightarrow \mu\epsilon$

$$c^2 = \frac{1}{\mu_0\epsilon_0}$$ now waves propagate

with $v = \frac{1}{\sqrt{\epsilon\mu}} = \frac{c}{n}$ $\quad \lambda = \lambda_0/n$ $\quad v \propto 1/\sqrt{\epsilon}$

of refraction $n = \sqrt{\frac{\epsilon\mu}{\epsilon_0\mu_0}} \approx \sqrt{\frac{\epsilon}{\epsilon_0}} = \sqrt{\epsilon_r}$

All previous formula carry over with the replacement $\epsilon_0 \to \epsilon$, $\mu_0 \to \mu$

Energy density $u = \frac{1}{2}\left(\epsilon E^2 + \frac{B^2}{\mu}\right)$

$$\vec{S} = \frac{1}{\mu}(\vec{E} \times \vec{B}) = (\vec{E} \times \vec{H})$$

$$I = \frac{1}{2}\epsilon v E_0^2$$

Turn to another ~~to~~ topic.

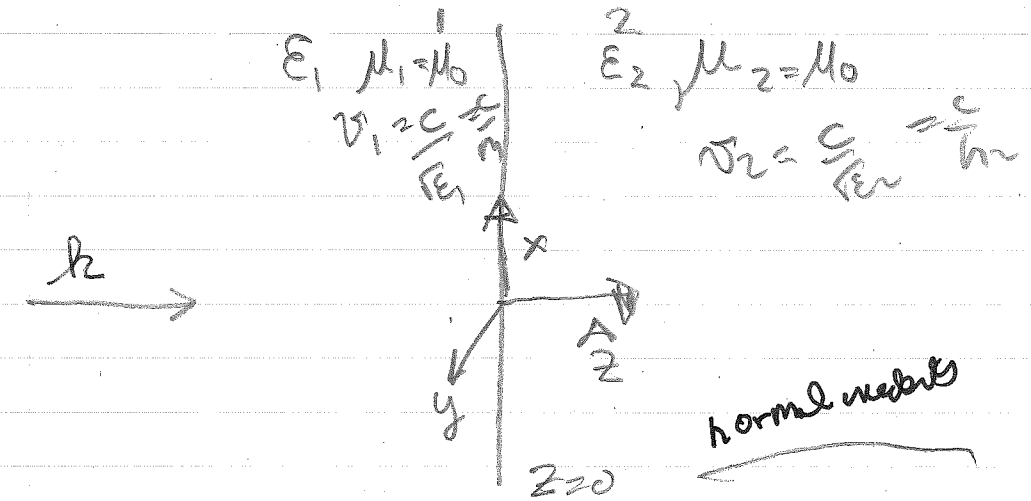
So far seen prop in a uniform medium. Let's now consider what happens when light hits boundary between two media – light passing from air to water or wine. ~~First~~ ~~Reflection & Transmission~~ ~~Normal~~

~~Consider an easy case~~

Incident.

Hae

What happens Suppose have Boundary.



Light comes in from left. What happens when hit boundary?

— recall string —

Part goes thru, part bounces back How much need to calculate

~~This~~ We'll do a special case first Normal incidence

Suppose light incident from left. pol in $\hat{x}$ direction

$$\vec{E}_I(z,t) = \tilde{E}_{0I}\, e^{i(kz-\omega t)}\, \hat{x}$$

Ask what is $\vec{B}_I$

$$\vec{B}_I(z,t) = \frac{\tilde{E}_{0I}}{v_1}\, e^{i(kz-\omega t)}\, \hat{y}$$

There will be reflected wave

$$\tilde{\vec{E}}_R(z,t) = \tilde{E}_{0R}\, e^{i(-k_1 z - \omega t)}\, \hat{x}$$

$$\tilde{\vec{B}}_R(z,t) = -\frac{\tilde{E}_{0R}}{v_1}\, e^{i(-k_1 z - \omega t)}\, \hat{y}$$

$\vec{k}_1 \uparrow z$

R

why same ω

In region 1 $\vec{E} = \vec{E}_I + \vec{E}_R$
$\vec{B} = \vec{B}_I + \vec{B}_R$

and transmitted wave

$$\tilde{\vec{E}}_T(z,t) = \tilde{E}_{0T}\, e^{i(k_2 z - \omega t)}\, \hat{x}$$

$$\tilde{\vec{B}}_T(z,t) = \frac{\tilde{E}_{0T}}{v_2}\, e^{i(k_2 z - \omega t)}\, \hat{y}$$

Goal get $\tilde{E}_{0R}$ $\tilde{E}_{0T}$ these are the only unknowns.

How do we do that?

Boundary conditions

Have no free charge or current at boundary

tangential compnt $\vec{E}$, normal compnt $\vec{D}$
& tangential compnt $\vec{H}$, " " of $\vec{B}$

now apply that idea

Note in medium ① total $\vec{E}, \vec{H}$ entire sum of incident + trans waves

$$I_I = \frac{1}{2}\epsilon_1 v_1 |\vec{E}_I|^2$$

Reflection coefficient

$$R = \frac{I_R}{I_I} = \frac{|E_{0R}|^2}{|E_{0I}|^2} = \left(\frac{(n_1-n_2)}{(n_1+n_2)}\right)^2$$

$$T = \frac{I_T}{I_I}$$ T = transmission coefficient

We expect $R+T=1$ nothing lost here no light

Then $T = 1 - \frac{(n_1-n_2)^2}{(n_1+n_2)^2}$

$$= \frac{(n_1+n_2)^2-(n_1-n_2)^2}{(n_1+n_2)^2} = \frac{4n_1n_2}{(n_1+n_2)^2}$$

Sanity check $n_1 = n_2$

Check with direct calculation

$$T = \frac{\epsilon_2 v_2}{\epsilon_1 v_1}\left|\frac{\vec{E}_T}{\vec{E}_I}\right|^2 = \frac{v_1}{v_2}\left(\frac{2n_1}{n_1+n_2}\right)^2$$

$$\frac{v_1}{v_2} \to \frac{n_2}{n_1}\left(\quad\right)$$

So agrees

$\vec{E}$ is entr cts tangential, so is $\vec{B} = \mu_0 \vec{H}$

$z=0$

$$\tilde{E}_{0I} + \tilde{E}_{0R} = \tilde{E}_{0T}$$

$$\frac{1}{v_1}\left(\tilde{E}_{0I} - \tilde{E}_{0R}\right) = \frac{1}{v_2}\tilde{E}_{0T}$$

two eqns 2 unknowns

$$\beta \equiv \frac{v_1}{v_2}$$

$$2\tilde{E}_{0I} = \tilde{E}_{0T}(1+\beta) \qquad v_{1,2} \propto \frac{1}{n_{1,2}}$$

$$\tilde{E}_{0T} = \frac{2}{1+\beta}\tilde{E}_{0I} = \frac{2v_2}{v_1+v_2} = \frac{2n_1}{n_1+n_2}$$

$$\tilde{E}_{0R} = \tilde{E}_{0T} - \tilde{E}_{0R} = \frac{2-(1+\beta)}{1+\beta}\tilde{E}_{0I}$$

$$\tilde{E}_{0R} = \frac{1-\beta}{1+\beta}\tilde{E}_{0I} = \frac{v_2-v_1}{v_1+v_2}\tilde{E}_{0I} = \frac{n_1-n_2}{n_1+n_2}$$

How to check — limit $\beta \to 1$

OK

if $v_2 > v_1$ R is in phase

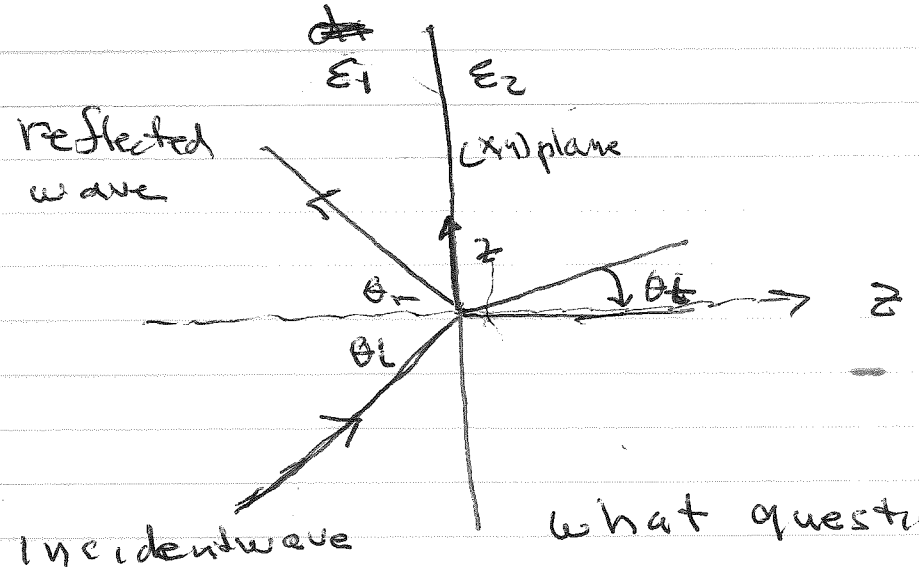
$v_1 > v_2$ Phase change

low dens → high d

Part This week's lecture will be on middle

Now we do the general case of oblique angle

Show demos first



what questions?

given θ_i, want θ_r, θ_t

also will want direction of $\vec{E}_{i,r,t}$

this gives polarization of light

Will use the boundary conditions
tangential E, tangent continuous
normal D, B continuous
First step enumerate field

Incident WF $\quad \vec{E}_I = \vec{E}_{I0}\, e^{i(\vec{k}_I\cdot\vec{r}-\omega t)}$

WF $\quad \vec{B}_I = \dfrac{\hat{k}_I \times \vec{E}_I}{v_1}$

reflected $\quad \vec{E}_R = \dfrac{\vec{E}_{R0}}{\cancel{v_1}}\, e^{i(\vec{k}_R\cdot\vec{r}-\omega t)}$

$\vec{B}_R = \dfrac{\hat{k}_R \times \vec{E}_R}{v_1}$

$\vec{E}_{I0}$, $\vec{E}_{R0}$, $\vec{E}_{T0}$ are complex ~~numbers~~ vectors

$\hat{k}_r$ not $\parallel$ to $\hat{k}_I$
have to figure out where it goes

Transmitted $\vec{E}_T = \vec{E}_{T_0} \, e^{i(\vec{k}_T \cdot \vec{r} - \omega t)}$

$$\vec{B}_T = \frac{\hat{k}_T \times \vec{E}_T}{v_2}$$

Notice the same $e^{-i\omega t}$ is everywhere
this is the only way to satisfy Bound Cond
advantage of complex notation. Get mag of $k_{I,R,T}$
so we may write $k_I = k_R = \frac{\omega}{c}\sqrt{\varepsilon_1} = \frac{\omega}{c} n_1$

$$k_T = \frac{\omega}{c} n_2$$

Can get a lot of info from mere existance of boundary conditions BC

The interface is at $z=0$

The ~~bou~~ BC must be satisfied at <u>all</u> values of x, y on the interface

Why? If not true BC won't be satisfied

This means at $z=0$

$$\vec{k}_I \cdot \vec{r} = \vec{k}_R \cdot \vec{r} = \vec{k}_T \cdot \vec{r}$$

$\vec{r}$ refers to position in complex plane
must be true at one place to satisfy BC. But that

place can be anywhere

Example from Griffiths

If above is true at one point changing $\vec{r}$ will make it not true

A B C a b c #6

Another way do model problem

$$A e^{iax} + B e^{ibx} = C e^{icx}$$ (like our problem with g = 0)

show

take $x=0$ $\quad A + B = C$ $\quad$ and $a=b=c$

so $\quad A\left(e^{iax} - e^{icx}\right) + B\left(e^{ibx} - e^{icx}\right) = 0$

Take $\frac{d}{dx}$ at $x=0$ $\quad A(a-c) = -B(b-c)$ (1)

Take $\frac{d^2}{dx^2}$ at $x=0$ $\quad A(a^2-c^2) = -B(b^2-c^2) =$ (2)

Take $\frac{(2)}{(1)}$ $\quad (a+c) = (b+c)$

$\Rightarrow a = b$

$A(a-c) = -B(a-c)$

either $c = a$ or $A = -B$

if $A = -B$ $\quad C = 0$ which violates give

so $\underline{c = a = b}$

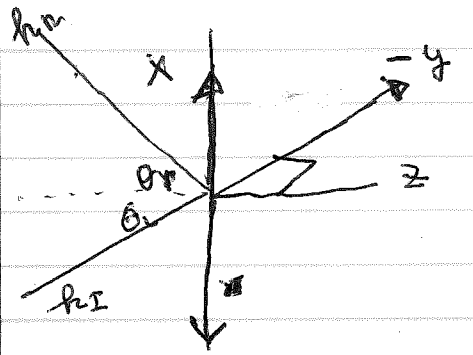
So use BC idea to relate I & R

$$x(k_I)_x + y(k_I)_y = x(k_R)_x + y(k_R)_y = x(k_T)_x + y k_{Ty}$$

can set $x=0$ $\quad (k_I)_y = (k_R)_y = (k_T)_y$

or $y=0$ $\quad (k_I)_x = k_{Rx} = k_{Tx}$

Look at interface plane



Side

Y axis ~~into~~ out of page k_I in x z plane, $k_{Iy}=0$

Thus $k_{Iy} = 0 = k_{Ry} = k_{Ty}$

can work in xz plane, <u>plane of incidence</u>

$\vec{k}_{I,R,T}$ in same plane just as drawn

Look at x comp $(k_I)_x = k_I \sin\theta_I$

$= (k_R)_y = k_I \sin\theta_R$

$\theta_I = \theta_R$ since angles go from 0 to $\pi/2$

angle of incidence = angle of reflection

also $k_I \sin\theta_I = k_T \sin\theta_T$

$\frac{\omega}{c} n_1 \sin\theta_T = \frac{\omega}{c} n_2 \sin\theta_T$

$$n_1 \sin\theta_I = n_2 \sin\theta_T$$ Snell's Law

These 2 results are true for any type of plane wave
sound
seismic
E & M

also marching band moving at angle to a boundary between low & high grass

But EM is much richer
$\vec{E}$ & $\vec{B}$ can have
x & y component

Let's be specific about BC. There are no free charges or free currents
tangential E is continuous
$\underline{w\,E}$
From Faraday

$$(\vec{E}_{0I} + \vec{E}_{0R})_{x,y} = (\vec{E}_{0T})_{x,y}$$

Normal D is continuous
From Coulomb/Gauss

$$\epsilon_1 [\vec{E}_{0I} + E_{0R}]_z = \epsilon_2 (E_{0T})_z$$

because we have no free charges on the dielectric surface

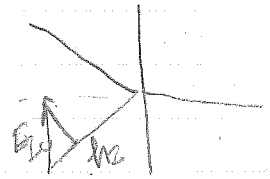
Why? no new info
Don't bother with $\vec{B}$'s given $\vec{E}$'s know $\vec{B}$

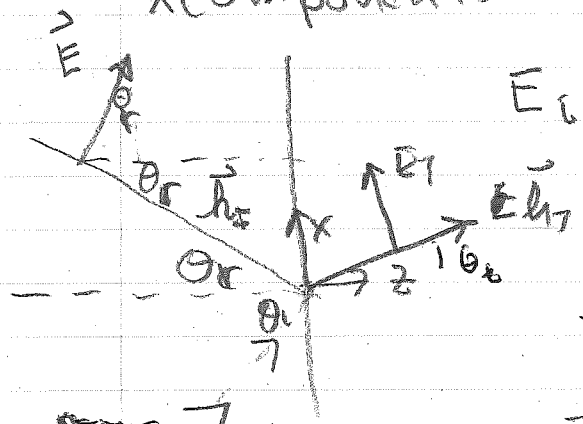
To proceed we need to specify polarization of light. There are 2 separate cases, depending on whether light pol $\perp$ or $\parallel$ to plane of incidence. Why different cases?

Physics is oscillating dipoles at surface $\vec{P} \propto \vec{E}$

Suppose incident light pol $\parallel$ to plane of incidence means $\vec{E}$ has only x, z components

x components



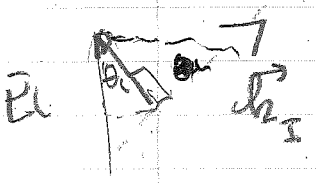


$$E_{io}\cos\theta_i + E_{ro}\cos\theta_r = E_{To}\cos\theta_T$$

z component of D

$$\epsilon_1\left[-E_{oi}\sin\theta_i + E_{or}\sin\theta_r\right] = \epsilon_2 E_{ot}\sin\theta_t$$

two eqns two unknowns



Again we have 2 eqns 2 unknowns

Simplify eqns whatever

define $\beta = n_2/n_1$

$$\alpha = \frac{\cos\theta_T}{\cos\theta_I}$$

Solve 2 eqns 2 unknowns

$$\boxed{\begin{aligned} E_{0R} &= \left(\frac{\alpha-\beta}{\alpha+\beta}\right) E_{0I} \\ E_{0T} &= \frac{2}{\alpha+\beta} E_{0I} \end{aligned}}$$

Fresnel formulae

Fresnel eqns

Transmitted wave in phase with incident wave

Reflected wave is in phase if $\alpha > \beta$

or 180° out of phase if $\alpha < \beta$

α & β depend on indices of refraction

β (obvious!)

Let's look at α

$$\alpha = \frac{\sqrt{1-\sin^2\theta_T}}{\cos\theta_I} = \frac{\sqrt{1-\left(\left(\frac{n_1}{n_2}\right)\sin\theta_I\right)^2}}{\cos\theta_I}$$

How can we check eqns

normal incidence $\theta_I = 0$, $\alpha = 1$

$\theta_I = 0 \quad \alpha = 1$

$\theta_I = \frac{\pi}{2}, \alpha = \infty$

get the previous results

$\tan \theta_B = \frac{n_2}{n_1}$

grazing incidence $\theta_I = 90°$

$\alpha \to \infty$ <u>entire</u> wave is <u>reflected</u>

$E_{0R} = E_{0I}$, $E_{0T} \to 0$

Can the reflected wave be eliminated

$\alpha = \beta$ means

$$\frac{\cos\theta_T}{\cos\theta_I} = \frac{n_2}{n_1} = \frac{\sqrt{1-\left(\frac{n_1}{n_2}\sin\theta_I\right)^2}}{\cos\theta_I} = \beta$$

$\frac{n_1}{n_2} = \beta$

when this eqn satisfied $\theta_I = \theta_B$ – Brewster angle

$$\frac{n_2^2}{n_1^2}\cos^2\theta_B = 1 - \frac{n_1^2}{n_2^2}\sin^2\theta_B$$

~~$\tan\theta_I$~~ =

$$\beta^2\cos^2\theta_B = 1 - \frac{1}{\beta^2}\sin^2\theta_B$$

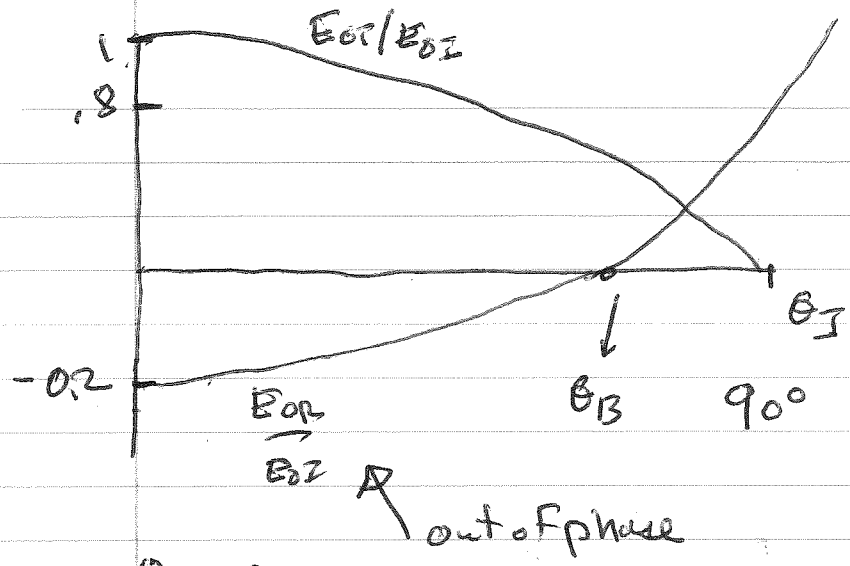
$$\beta^2 - \beta^2\sin^2\theta_B = 1 - \frac{1}{\beta^2}\sin^2\theta_B$$

$$\frac{+\beta^2 - 1}{+\beta^2 - \frac{1}{\beta^2}} = \sin^2\theta_B = \frac{\beta^4 - \beta^2}{\beta^4 - 1} = \frac{\beta^2}{1+\beta^2}, \quad \tan\theta_B = \beta$$

$$\beta^2 - 1 < - \frac{1}{\beta^2}$$

$\beta = \frac{n_2}{n_1} > 1$

• from air $n_1 = 1$ to glass $n_2 = 1.5$



$\beta = 1.5$

$\theta_I = 0$

$$I_I = \frac{1}{2} \epsilon_1 v_1 E_{0I}^2 \cos\theta_I$$

Power per unit area striking boundary

$\vec{S} \cdot \hat{n}$

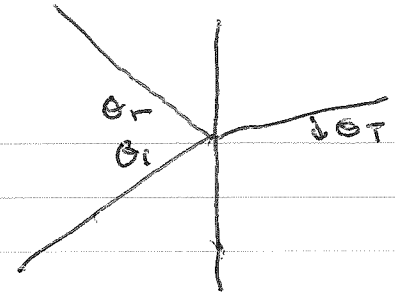
$$R = \frac{I_R}{I_I} = \left|\frac{E_{0R}}{E_{0I}}\right|^2 = \left(\frac{\alpha - \beta}{\alpha + \beta}\right)^2$$

T = more complicated

$$= \frac{I_T}{I_I} = \frac{v_2}{v_1} \left|\frac{E_{0T}}{E_{0I}}\right|^2 \frac{\cos\theta_T}{\cos\theta_I} = \frac{\alpha\beta\, 4}{(\alpha+\beta)^2}$$

$R + T = 1$

Won't do $\perp$ case - more extension for possible ?



$$\vec{E}_I = E_{I0}\,\hat{y}\; e^{i(k_I \cdot r - \omega t)}$$

$$\vec{B}_I = \frac{\hat{k}_I \times \vec{E}_I}{v_1} \qquad \hat{k}_I = \sin\theta_i\,\hat{x} + \cos\theta_i\,\hat{z}$$

$$\vec{E}_R = E_{R0}\,\hat{y}\; e^{i(k_R \cdot \vec{r} - \omega t)}$$

$$\vec{E}_T = E_{T0}\,\hat{y}\; e^{i(k_T \cdot \vec{r} - \omega t)}$$

BC Tangential E continuous

$$E_{I0} + E_{R0} = E_{T0}$$

Normal $\vec{D}$ is continuous

$\vec{D} = \epsilon_1 \vec{E}$ or $\epsilon_2 \vec{E}$ has no $\sin\theta = 0$

normal component need other eq.

BC normal $\vec{B}$