Finally electromagnetic waves
(in vaccuum)

Maxwell eqns

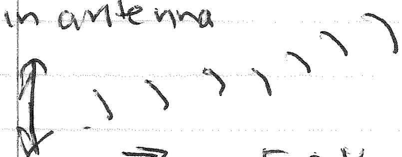
$\vec{\nabla}\cdot\vec{E} = 0$ , $\vec{\nabla}\times\vec{E} = -\frac{\partial \vec{B}}{\partial t}$ Faraday

$\vec{\nabla}\cdot\vec{B} = 0$ $\vec{\nabla}\times\vec{B} = \mu_0\epsilon_0\frac{\partial \vec{E}}{\partial t}$ Ampere Maxwell
AM

Why not $\vec{E}, \vec{B} = 0$ ?

Somewhere $\uparrow I(t)$ $\rightarrow v(t)$ accelerating charges

ac in antenna



$\rightarrow$ E & M waves
creation of waves in 323

So assume $\vec{E} \neq 0$ , $\vec{B} \neq 0$

what do M.Eqns tell us $B(t) \rightarrow E(t)$ Faraday
$E(t) \rightarrow B(t)$ AM
$\rightarrow E(t)$ F
$\rightarrow B(t)$ AM

Goal: examine Maxwell eqns
show $\vec{E}$ & $\vec{B}$ each satisfy wave equation

How to proceed?

$$\vec{\nabla}\times(\vec{\nabla}\times\vec{E}) = -\vec{\nabla}\times\frac{\partial \vec{B}}{\partial t} = -\frac{\partial}{\partial t}\vec{\nabla}\times\vec{B} = -\frac{\partial^2}{\partial t^2}\mu_0\epsilon_0\vec{E}$$

$$\vec{\nabla}\times(\vec{\nabla}\times\vec{E}) = \vec{\nabla}(\vec{\nabla}\cdot\vec{E}) - \nabla^2\vec{E}$$
$$= -\nabla^2 E$$

Proof on next page

$$\left(\nabla \times (\vec{\nabla} \times \vec{E})\right)_k =$$

$$= \epsilon_{ink} \frac{\partial}{\partial x_i} \left( \frac{\partial}{\partial x_j} E_m \right) \epsilon_{jmn}$$

Use $\epsilon_{ink}\, \epsilon_{jmn} = \epsilon_{kin}\, \epsilon_{jmn} = \delta_{kj}\delta_{im} - \delta_{km}\delta_{ij}$

so $\left(\vec{\nabla} \times (\vec{\nabla} \times \vec{E})\right)_k$

$$= \frac{\partial}{\partial x_i} \frac{\partial}{\partial x_k} E_i - \frac{\partial \partial E_k}{\partial x \partial x}$$

$$= \nabla_k \vec{\nabla} \cdot \vec{E} - \nabla^2 E_k$$

So we have $\boxed{\nabla^2 \vec{E} = \mu_0 \epsilon_0 \frac{\partial^2 \vec{E}}{\partial t^2}}$ each component of $\vec{E}$ satisfies wave eq speed $= c$

Similarly take $\vec{\nabla} \times (\vec{\nabla} \times \vec{B}) = \mu_0 \epsilon_0 \frac{\partial}{\partial t}(\vec{\nabla} \times \vec{E})$

to get
$$\nabla^2 \vec{B} = \mu_0 \epsilon \frac{\partial^2 \vec{B}}{\partial t^2}$$

Each equation represents 3 equation

Each component of $\vec{E}$ & $\vec{B}$ obeys wave eq

We need to deal with all 3 directions

In space
$$\mu_0 \epsilon_0 = \frac{1}{c^2}$$
$c$ = speed of light (in vacuum) $= 3 \times 10^8$ m/s

Look for plane wave solutions

$$\vec{E}(\vec{r}, t) = \vec{E}_0 e^{i(\vec{k} \cdot \vec{r} - \omega t)}, \quad \vec{B} = \vec{B}_0 e^{i(\vec{k} \cdot \vec{r} - \omega t)}$$
Why same

Constant $\vec{E}_0$ is complex valued so is $\vec{B}_0$

Take real part at end of calculation

$\vec{k}$ is wave vector, gives direction of propagation

$$\vec{k} \cdot \vec{r} = k_x x + k_y y + k_z z$$

line z-axis with $\vec{k}$

To get a simpler picture $\vec{k} = k \hat{z}$

$\vec{k} \cdot \vec{r} = kz$ wave repeats if $z \to z + \frac{2\pi}{k} = z + \lambda$ wave repeats plane wave all points on plane $\perp$ to $z$ axis have same phase

Go back to general case

$\vec{\nabla} \cdot \vec{E} = 0 \rightarrow$ ?

$$\vec{\nabla} = \hat{x}\frac{\partial}{\partial x} + \hat{y}\frac{\partial}{\partial y} + \hat{z}\frac{\partial}{\partial z}$$

$$\vec{\nabla}\, e^{i\vec{k}\cdot\vec{r}} = e^{i(k_x x + k_y y + k_z z)}$$

$$= i\left[\hat{x}\, k_x + \hat{y}\, k_y + \hat{z}\, k_z\right] e^{i\vec{k}\cdot\vec{r}}$$

$$= i\vec{k}\, e^{i\vec{k}\cdot\vec{r}}$$

Thus $\vec{k}\cdot\vec{E}_0 = 0$ direction of $\vec{E} \perp \vec{k}$

$\vec{E} \perp$ propagation direction.

Similarly $\vec{\nabla}\cdot\vec{B} = 0 \Rightarrow \vec{k}\cdot\vec{B}_0 = 0$

$\vec{E}$ & $\vec{B}$ line in plane $\perp$ $\vec{k}$

EM waves = <u>Transverse</u> waves

What about $\nabla^2 \vec{E}$ ? & $\frac{\partial^2 \vec{E}}{\partial t^2}$ ?

$$= -k^2 \vec{E}$$

Thus we have $\frac{\partial^2 \vec{E}}{\partial t^2} = -\omega^2 \vec{E}$

$$k^2 \vec{E}(\vec{r},t) = \frac{\omega^2}{c^2}\vec{E}(\vec{r},t)$$

$|\vec{k}| = k = \frac{\omega}{c}$ take positive root direction in $\hat{k}$

Now relate $\vec{E}$ to $\vec{B}$ in magnitude & direction

$$\vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$$

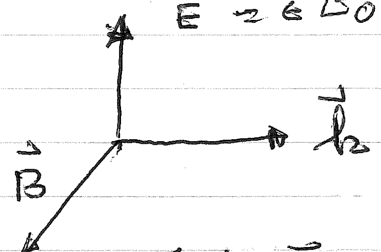
$$i\vec{k} \times \vec{E} = i\omega \vec{B}$$

$$\vec{B} = \frac{\vec{k}}{\omega} \times \vec{E} \quad , \quad \vec{B} \cdot \vec{E} = 0$$

$\vec{B}$ & $\vec{E}$ have <u>same</u> phase

Magnitude $|\vec{B}| = |\vec{E}|/c$ SI units

$\vec{E} = \hat{e} E_0$




direction of B

direction of $\vec{E}$ defines polarization direction

$\vec{E}, \vec{B}, \vec{k}$ form right-handed coordinate system

Why plane wave? general sol'n is a linear superposition of plane waves

Fourier's theorem

But make no mistake - there is something remarkable here waves travel and infinite distance very different than $\vec{E} \sim \frac{1}{r}$, $\vec{B} \sim \frac{1}{r^2}$

all caused by <u>time dependence</u>

Compute energy flux, energy density and momentum density

$|\vec{S}|$ = Intensity = $\frac{\text{Power}}{\text{Area}}$

Energy flow density $\vec{S} = \frac{1}{\mu_0}(\vec{E} \times \vec{B})$

we have something that is quadratic in the field, So we must take $\mathrm{Re}\,\vec{E} \times \mathrm{Re}\,\vec{B}$

$$= \tilde{E}_0 e^{i(\vec{k}\cdot\vec{r}-\omega t)}$$

Take real $\vec{E} = E_0 \cos(\vec{k}\cdot\vec{r}-\omega t+\delta)\,\hat{\epsilon}$

where $\hat{\epsilon} \perp \vec{k}$ $\qquad \hat{\epsilon}\cdot\vec{k} = 0$

$$\vec{B} = \frac{\vec{k}}{\omega} \times \vec{E}$$

$$= \frac{\vec{k}\times\hat{\epsilon}}{\omega} E_0 \cos(\vec{k}\cdot\vec{r}-\omega t+\delta)$$

Direction of $\vec{E}\times\vec{B}$

$$\hat{\epsilon} \times (\vec{k}\times\hat{\epsilon}) = \vec{k}\,\hat{\epsilon}\cdot\hat{\epsilon} - \hat{\epsilon}\,\vec{k}\cdot\hat{\epsilon}$$

$$= \vec{k}$$

Poynting vector points in direction of propagation

$$\vec{S} = \frac{\hat{k}}{c}\frac{1}{\mu_0} E_0^2 \cos^2(\vec{k}\cdot\vec{r}-\omega t+\delta)$$

Now $\omega$ is very large $10^{15}\ s^{-1}$ visible light

see only average value

Take a fixed position in space & time ave

$$\langle|\vec{S}|\rangle = \frac{E_0^2}{c\mu_0} \frac{1}{T}\int_0^T dt\, \cos^2(-\omega t+\phi)$$

$T = 2\pi/\omega$ $\qquad \phi = \vec{k}\cdot\vec{r}-\omega t$

The result

$$\boxed{\langle \vec{S} \rangle = \frac{\hat{k}}{c}\frac{1}{\mu_0}\frac{E_0^2}{2}} = \frac{\hat{k}\, c\epsilon_0 E_0^2}{2} = I$$

Intensity

average Power/area

Proof

$$\cos^2 x = \frac{1+\cos 2x}{2}$$

$$\frac{1}{T}\int_0^{2\pi/\omega} dt\ \cos^2(-\omega t+\phi) = \frac{1}{T}\int_0^{T} dt\ \frac{1+\cos 2(-\omega t+\phi)}{2}$$

$$= \frac{T}{2T} + \frac{1}{2T}\left(-\frac{1}{2\omega}\right)\sin(-2\omega t+\phi)\Big|_0^{\frac{2\pi}{\omega}}$$

$$= \frac{T}{2T} = \frac{1}{2}$$

Trick can compute time average of $\vec{S}$

by computing $\frac{1}{2}\frac{1}{\mu_0}\mathrm{Re}\,(\vec{E}\times\vec{B}^*) = \langle \vec{S} \rangle$

here all phases in E & B cancel

but expression is more general

Get energy density

$$u = \frac{1}{2}\left(\epsilon_0 E^2 + \frac{B^2}{\mu_0}\right) \qquad \frac{1}{c^2} = \mu_0\epsilon$$

$$|\vec{B}| = \frac{|\vec{E}|}{c}, \quad \frac{B^2}{\mu_0} = \frac{1}{c^2\mu_0}E^2 = \epsilon_0 E^2$$

Magnetic contribution = electric contribution

$$u = \epsilon_0 E^2$$

$$\langle u \rangle = \frac{1}{2} \epsilon_0 \,\mathrm{Re}\, \vec{E} \cdot \vec{E}^*$$

$$= \frac{1}{2} \epsilon_0 \vec{E} \cdot \vec{E}^*$$

Momentum density

$$\vec{p} = \frac{\vec{S}}{c^2}$$

$$\langle \vec{p} \rangle = \frac{\hat{k}\, e}{c^3 \mu_0} \frac{E_0^2}{2} \qquad c^2 = \frac{1}{\mu_0 \epsilon}$$

$$= \frac{\hat{k}}{c} \frac{\epsilon_0 E_0^2}{2}$$

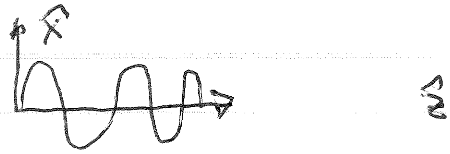
$$|\langle \vec{p} \rangle| = \frac{\langle u \rangle}{c}$$

As Expected from photons

## Polarization of transverse waves

Waves propagating in $\hat{z}$ direction

But there are 2 directions $\perp$ to



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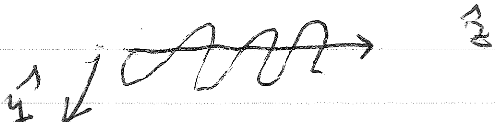


Direction of transverse displacement = polarization vector $\hat{n}$

$$\hat{n} = \cos\theta\,\hat{x} + \sin\theta\,\hat{y}$$ plane of displacement

general scalar wave

$$\tilde{\vec{f}} = \tilde{A}\cos\theta\, e^{i(kz-\omega t)}\,\hat{x} + \tilde{A}\sin\theta\, e^{i(kz-\omega t)} e^{i\phi}\,\hat{y}$$

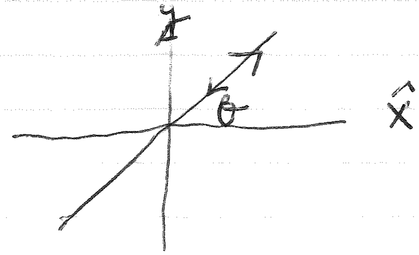
e.g. $\phi$ is a phase difference

- propagation factor $e^{i(kz-\omega t)}$ is same

if $\phi = 0, 2\pi, 4\pi \ldots$

The $\tilde{\vec{f}} = \tilde{A}\, e^{i(kz-\omega t)}\,[\hat{n}]$

wave coming towards you at fixed z



displacement moves on a line

linear polarization

An other interesting case

$$\phi = \pm \pi/2$$

$$\vec{f} = \tilde{A}\, e^{i(kz-\omega t)} \left[\hat{x} \pm i\hat{y}\right]$$

Take fixed $z=0$, real part, $\tilde{A}$ = real

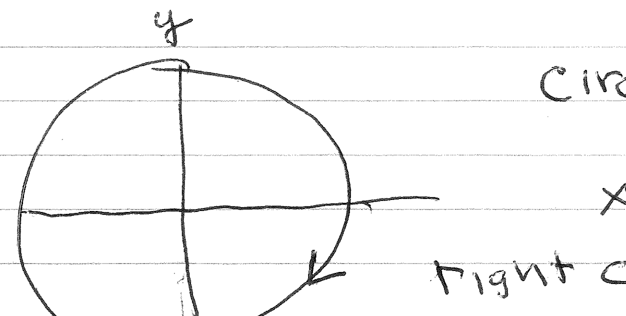
$$\frac{\mathrm{Re}\,\vec{f}}{A} = \mathrm{Re}\left(e^{-i\omega t}\hat{x} + e^{-i\omega t \pm \frac{i\pi}{2}}\hat{y}\right)$$

$$= \cos\omega t\,\hat{x} + \cos\left(\omega t \pm \frac{\pi}{2}\right)$$

$$= \cos\omega t\,\hat{x} \mp \sin\omega t\,\hat{y}$$

$$\left|\frac{\mathrm{Re}\,\vec{f}}{A}\right|^2 = \cos^2\omega t + \sin^2\omega t = 1$$

fixed mag but rotation

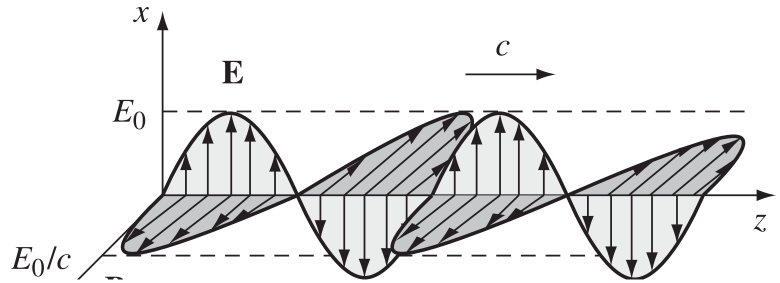



Circular polarization

using − sign

right circ. pol

using + arrow points the other way
left circ polarization




Ordinary light beams are produced by different radiating atoms.
The electric fields are randomly oriented. The light is un-polarized.

Polarizing sheets allow light that passes thru to be polarized
A relevant Tube is
https://www.youtube.com/watch?v=9In9E5fT__E

Polarizers have long strips of molecules that vibrate in response to electric fields in a specified direction. The energy of those waves is given up to the molecules.
Other light waves with E in orthogonal direction gets through and is is polarized

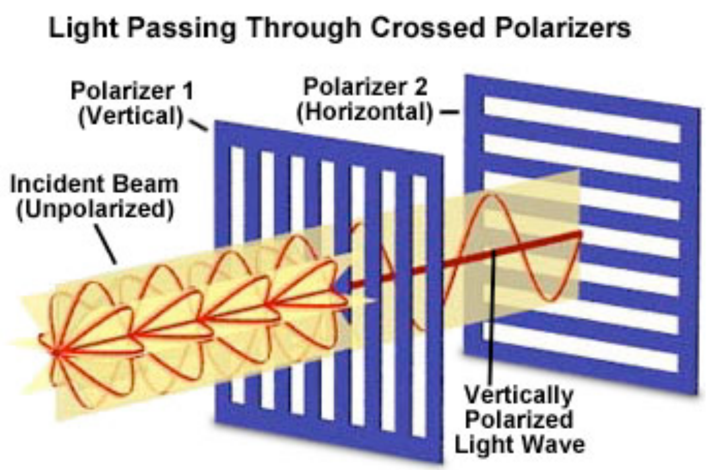
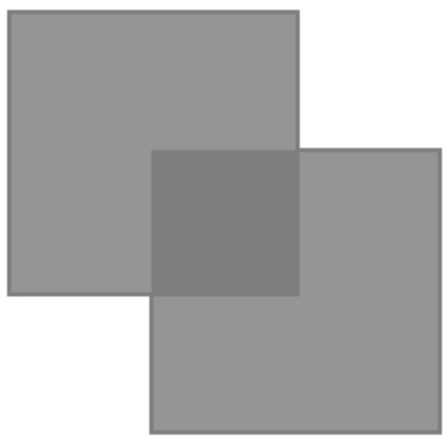
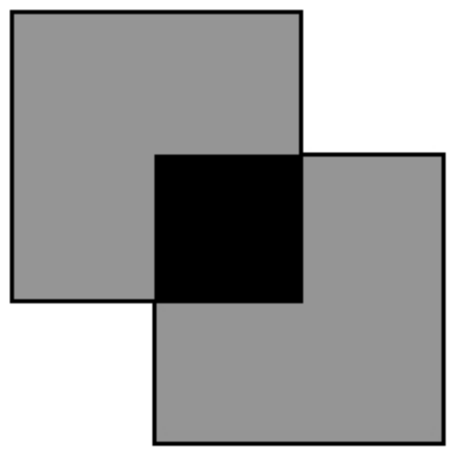



Figure 1

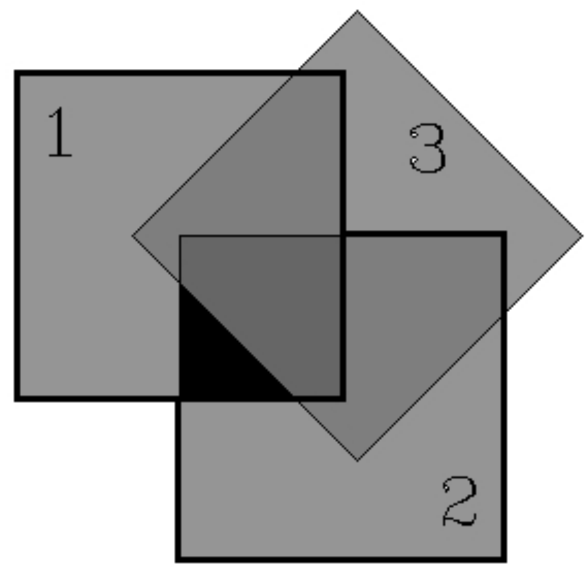
# Demonstration I'd have shown



Parallel axes.



Crossed axes.




Polarizer (3) between two crossed polarizers (1) and (2).

Maxwell's eq - partial diff eqns

to solve need BC

<u>Boundary Conditions</u> needed for interactions between light to matter

1

2

In general E, B, D, H not continuous at surface that carries $\sigma$ or $\vec{K}$

closed surface

$$\vec{\nabla} \cdot \vec{D} = \rho_f \rightarrow \oint_{\mathcal{S}} \vec{D} \cdot d\vec{a} = Q_{f\,enc} \quad (1)$$

$$\vec{\nabla} \cdot \vec{B} = 0 \rightarrow \oint_{\mathcal{S}} \vec{B} \cdot d\vec{a} = 0 \quad (2)$$

Stoke's theorem

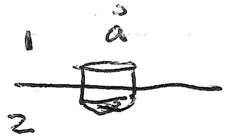
$$\vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} \rightarrow \oint_{P} \vec{E} \cdot d\vec{l} = -\frac{d}{dt}\int_{S} \vec{B} \cdot d\vec{a} \quad (3)$$

where any surface $\mathcal{S}$ bounded by closed loop P

$$\nabla \times \vec{H} = \vec{J}_f + \frac{\partial \vec{D}}{\partial t} \Rightarrow \oint \vec{H} \cdot d\vec{l} = I_f + \frac{d}{dt}\int_{S} \vec{D} \cdot d\vec{a} \quad (4)$$

Use eq 1 to get BC ~~on~~ for D

Gaussian pill box sides don't cont.




$$(\vec{D}_1 - \vec{D}_2) \cdot \vec{a} = \sigma_f\, a$$

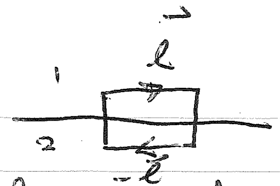
$$\rightarrow D_1^{\perp} - D_2^{\perp} = \sigma_f$$

$^{\perp}$ means $\perp$ to $\vec{a}$

Similarly (2) $\rightarrow$ $B_1^{\perp} - B_2^{\perp} = 0$

side view

For (3)




sides $\to 0$

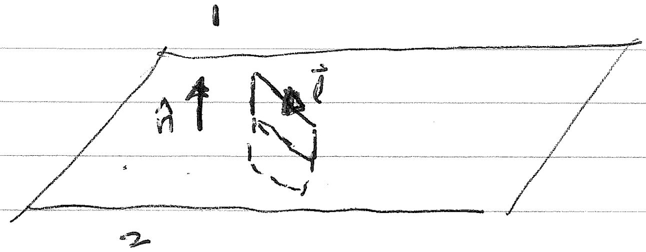
area $\to 0$

$$\vec{E}_1 \cdot \vec{l} - \vec{E}_2 \cdot \vec{l} = -\frac{d}{dt} \int \vec{B} \cdot da \to 0$$

$$E_1^{\parallel} - E_2^{\parallel} = 0$$

For 4 $\quad \vec{H}_1 \cdot \vec{l} - \vec{H}_2 \cdot \vec{l} = I_{free}$

Need more than side view




$I_{free}$ goes along surface no volume current

$I_{free}$ is total current thru loop

need current thru loop magnitude is $K_f l$

direction is $\hat{n} \times \vec{l}$ $\quad \vec{I}_{free} = \vec{K}_f \cdot (\hat{n} \times \vec{l})$

$$= (\vec{K}_f \times \hat{n}) \cdot \vec{l}$$

so $\boxed{\vec{H}_1^{\parallel} - H_2^{\parallel} = \vec{K}_f \times \hat{n}}$