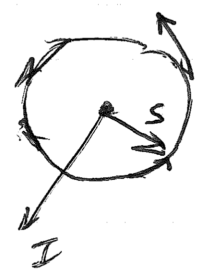
$$\vec{\nabla} \times \vec{B} = \mu_0 \vec{J}$$

Last time $\oint_C \vec{B} \cdot d\vec{l} = \mu_0 I$ (thru area bounded by C)

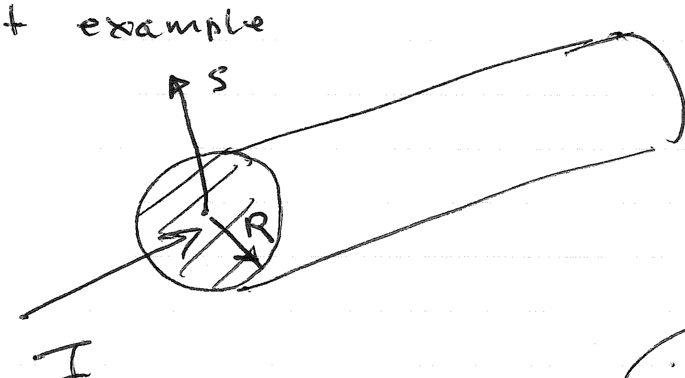
Example

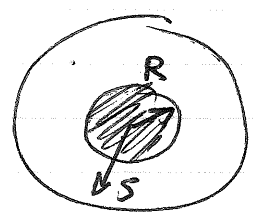


$$\vec{B} = \frac{\mu_0 I}{2\pi s} \hat{\phi}$$

$$\oint \vec{B} \cdot d\vec{l} = \oint \vec{B} \cdot dl\, \hat{\phi} = \oint B\, dl = 2\pi s B = \mu_0 I$$

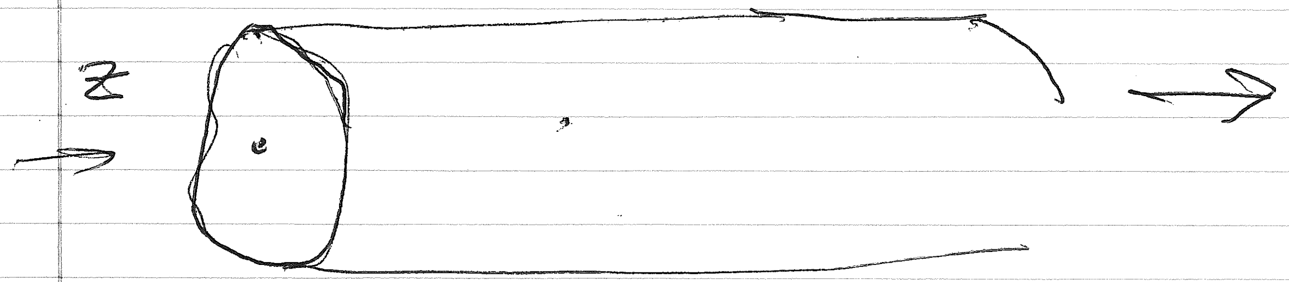
next example





Ampere's Law example

Long cylindrical conductor
Radius R

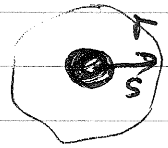


Current is uniform density $J = I/\text{Area} = I/(\pi R^2)$

Compute B inside and outside

$\vec{B}$ must be azimuthal $\vec{B} = B(s)\,\hat{\phi}$

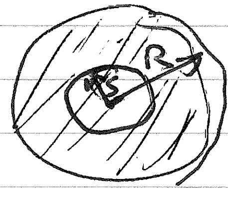
outside $\oint \vec{B}\cdot d\vec{l} = \mu_0 I \;\left(= \mu_0 \int \vec{J}\cdot d\vec{a}\right)$ current enclosed is I



$$B\,2\pi s = \mu_0 I$$

$$B = \frac{\mu_0 I}{2\pi s}$$

INSIDE



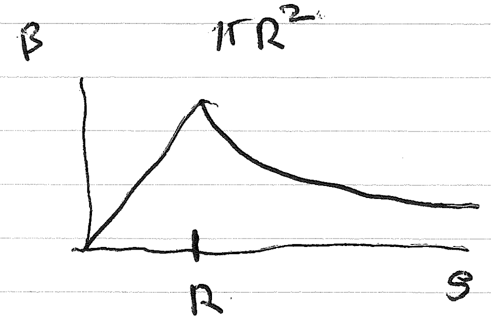
consider inside area defined by radius s
area $= \pi s^2$

B is still azimuthal

$$B\,2\pi s = \mu_0 \int \vec{J}\,da = \mu_0 \frac{I}{\pi R^2}\pi s^2$$

$$B = \frac{\mu_0 I}{2\pi}\frac{s}{R^2}$$

22,



More on $\vec{A}$

So far only used $\vec{A}$ as ~~easy~~ way to get ~~$\vec{\nabla}\cdot\vec{A}$~~ $\vec{\nabla}\cdot\vec{B} = 0$

$$\vec{B} = \vec{\nabla} \times \vec{A}$$

$\vec{A}$ is not unique

Can shift $\vec{A}$ by a vector

$$\vec{A}' = \vec{A} + \vec{\nabla}\Lambda(\vec{r})$$

$\Lambda(\vec{r})$ ← a scalar function

$$\vec{\nabla} \times \vec{A}' = \vec{\nabla} \times \vec{A} + \underbrace{\vec{\nabla} \times \vec{\nabla}\Lambda}_{0}$$

$\vec{B}$ is invariant under a **local gauge** transformation

Modern theories of fundamental theories are built ~~and~~ using the idea of invariance under local gauge transformation

$\vec{A}$ is not unique but

$\oint \vec{A}\cdot d\vec{l}$ is unique

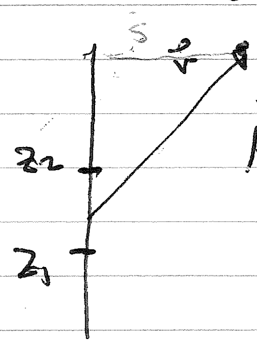
$$\oint_C \vec{A}\cdot d\vec{l} = \int_{\text{area}} (\nabla\times\vec{A})\cdot d\vec{S} = \int \vec{B}\cdot d\vec{S}$$

$= \Phi$ magnetic flux is unique

B is flux density

<u>Example</u>

$\vec{A}$ due to finite length of current in long straw



$$\vec{A}(\vec{r}) = \frac{\mu_0}{4\pi} I \hat{z} \int_{z_1}^{z_2} dz' \frac{1}{|\vec{r}-\vec{r}'|}$$

$\vec{r}' = (0, 0, z')$

$\vec{r} = (x, y, z)$

$$|\vec{r}-\vec{r}'| = \sqrt{s^2 + (z-z')^2}$$

$$s^2 = x^2 + y^2$$

$$\vec{A}(\vec{r}) = \frac{\mu_0}{4\pi} I \hat{h} \int_{z_1}^{z_2} dz' \frac{1}{\sqrt{s^2 + (z-z')^2}}$$

HW Length of wire $L \to \infty$ integrate on z' from 0 to L

What happens to $\vec{A}$ if $s \gg z_1, z_2, z$

& z_2

& $z \gg s, z_1, z_2$

$$= \frac{\mu_0}{4\pi} I \hat{h} \frac{1}{2} \left[\operatorname{ArcTanh} \frac{z_2 - z}{\sqrt{s^2 + (z_2 - z)^2}} - \tanh^{-1} \frac{z_1 - z}{\sqrt{s^2 + (z_1 - z)^2}} \right]$$

1B

$$\vec{A} = A(s,z)\,\hat{z}$$

$$\vec{B} = \vec{\nabla} \times \vec{A} = -\hat{\phi}\,\frac{\partial}{\partial s} A(s,z)$$ (G front cover)

gives $\vec{B}$ for finite length wire

there is z dependence

Partial Differential Eq for $\vec{A}$

More on vector potential $\vec{B} = \vec{\nabla} \times \vec{A}$

$$\boxed{\vec{A}(\vec{r}) = \frac{\mu_0}{4\pi} \int d^3r' \, \frac{\vec{J}(\vec{r}\,')}{|\vec{r} - \vec{r}\,'|}}$$

3 equations

solve by analogy with electrostatics ~~V is no change~~

$$\nabla^2 V = -\rho/\epsilon_0$$

$$V(\vec{r}) = \frac{1}{4\pi\epsilon_0} \int d^3r' \, \frac{1}{|\vec{r} - \vec{r}\,'|} \, \rho(\vec{r}\,')$$

$$A_x(\vec{r}) = \frac{\mu_0}{4\pi} \int d^3r' \, \frac{J_x(\vec{r}\,')}{|\vec{r} - \vec{r}\,'|}$$

$$\nabla^2 A_x(\vec{r}) = \mu_0 J_x(\vec{r})$$

$$\boxed{\nabla^2 \vec{A} = -\mu_0 \vec{J}}$$

Alternate Derivation

$$\nabla^2 \frac{1}{|\vec{r} - \vec{r}\,'|} = -4\pi \delta(\vec{r} - \vec{r}\,')$$

already done

$\vec{A}$ is not unique can add → Gauge transformation

define $\vec{A}\,' = \vec{A} + \vec{\nabla} \Lambda(\vec{r})$

does this change $\vec{B}$?

$$\vec{\nabla} \times \vec{A}\,' = \vec{\nabla} \times (\vec{A} + \vec{\nabla} \Lambda) = \vec{\nabla} \times \vec{A}$$

Jan 15

Summary of Maxwell eqns

Electrostatics Magnetostatics

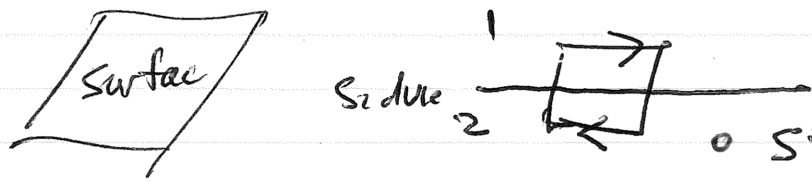
1 $\vec{\nabla}\cdot\vec{E} = \rho/\epsilon_0$ $\qquad$ $\vec{\nabla}\cdot\vec{B} = 0$ $\qquad$ $\vec{\nabla}\cdot\vec{J} + \frac{\partial\rho}{\partial t} = 0$

2 $\vec{\nabla}\times\vec{E} = 0$ $\qquad$ $\vec{\nabla}\times\vec{B} = \mu_0\vec{J}$

3 $\vec{\nabla}\cdot\vec{D} = \rho_{free}$ $\qquad$ $\rho = \rho_{free} + \rho_{bond}$

Electrostatics

• Boundary condition $\qquad$ tangential component of $\vec{E}$ continue



• Stokes theorem

which eq gave BC 2

For D Gaussian Pillbox

$$\int d^3r\, \vec{\nabla}\cdot\vec{D} = Q_{free}$$

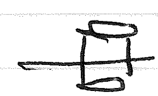
$$= \int d\vec{S}\cdot\vec{D} = Q_{free} \qquad d\vec{S} = \uparrow \quad \hat{n} = \uparrow$$

$$dA\,(\vec{D}_1 - \vec{D}_2)\cdot\hat{n} = \sigma_{free}\, dA$$

$$\therefore (\vec{D}_1 - \vec{D}_2)\cdot\hat{n} = \sigma_{free}$$ Divergence theorem

Now do same for $\vec{B}$

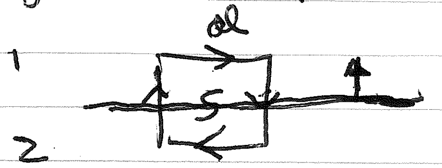
from $\vec{\nabla}\cdot\vec{B}$ Gaussian pillbox



$$\int d^3r\, \vec{\nabla}\cdot\vec{B} = \int d\vec{S}\cdot\vec{B} = 0$$

$$\boxed{(\vec{B}_1 - \vec{B}_2)\cdot\hat{n} = 0}$$

Tangential Comp of $\vec{B}$ use $\vec{\nabla} \times \vec{B} = \mu_0 \vec{J}$



squeeze area to small

$$\oint_S \vec{\nabla} \times \vec{B} \cdot d\vec{S} = \oint \vec{B} \cdot d\vec{l} = \mu_0 I(\text{enclosed})$$

Direction matters!

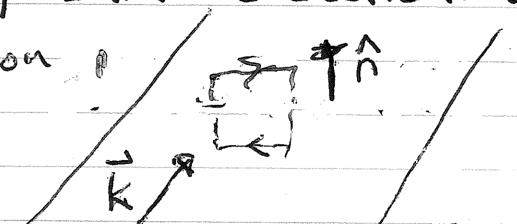
Here $= (\vec{B}_1 - \vec{B}_2) \cdot d\vec{l} = \mu_0 \, dI_{encl}$

$= \mu_0 \vec{K} \cdot d\vec{l}_\perp$

only surface current contributes

Now direction

1 = above



If $d\vec{l} \parallel \vec{K}$, $B \cdot dl$ is continuous

the compont of $\vec{B} \perp \vec{K}$ is discontinuous

~~2 equations~~

$$\boxed{(\vec{B}_1 - \vec{B}_2) = \mu_0 \vec{K} \times \hat{n}}$$

These boundary condition are important in magnetic materials

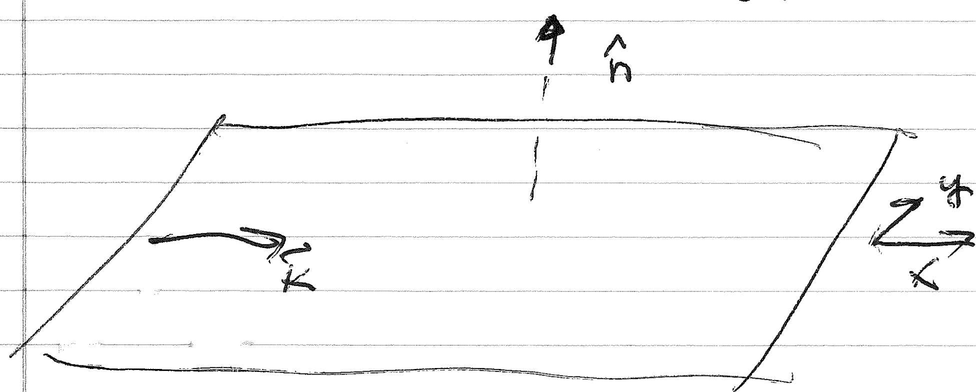
Right now electrostatic relations more complicated than magnetostatic — effects of medium into acount.

Need to do same for magnetism

PINASSport

Example

Currents ~~flat~~ on flat she



$$\vec{B}_{above} = -\frac{\mu_0 K}{2}\hat{y}$$

$$\vec{B}_{below} = \frac{\mu_0 k}{2}\hat{y}$$

Check that

$$\vec{B}_{above} - \vec{B}_{below} = \mu_0 \vec{K} \times \hat{n} \quad , \hat{n} = \hat{z}$$

$$-\mu_0 K \hat{y} \stackrel{?}{=} \mu_0 K \hat{i} \times \hat{z}$$

$$= -\mu_0 K \hat{y}$$