Conservation of energy

Maxwell eqns

$\rho = \rho_f + \rho_b$

$J = J_f + J_b + J_p$

$$\vec{\nabla}\cdot\vec{E} = \rho/\epsilon_0 \qquad \vec{\nabla}\times\vec{E} = -\partial\vec{B}/\partial t$$

$$\vec{\nabla}\cdot\vec{B} = 0 \qquad \vec{\nabla}\times\vec{B} = \mu_0\vec{J} + \mu_0\epsilon_0\frac{\partial\vec{E}}{\partial t}$$

This along with

$\vec{F} = q(\vec{E} + \vec{v}\times\vec{B})$ tells all

about electrodynamics

coupled equations - complicated

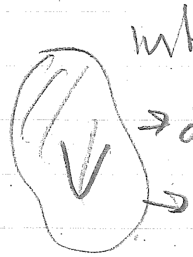
Need symmetries and conservation laws

example current conservation

$\vec{J}$ = flux of current

$$\vec{\nabla}\cdot\vec{J} + \frac{\partial\rho}{\partial t} = 0$$ local

inte



$d\vec{a}$

S

$$\oint \vec{J}\,d\vec{a} + \frac{dQ_{in}}{dt} = 0$$ global

$$\frac{dQ_{out}}{dt} + \frac{dQ_{in}}{dt} = 0$$

~~Do~~ Goal find similar equation for energy

Energy flow between fields & charges

Suppose there's a region in which $\vec{E}(\vec{r},t)$, $\vec{B}(\vec{r},t)$ exist & there are charged particles that can move

Let's first step see how <u>charges</u> get energy

$\vec{E}, \vec{B} \neq 0$

Consider a region and a bit of charge $\{dq\}$ $dq = \rho\, d\tau$ ↙ volume element

Such a bit of matter experiences forces $\vec{F}$ that move charges thru displacement $d\vec{l}$ · work is done

dW_{part} (work on <u>part</u>icle)

$$dW_{part} = \vec{F}\cdot d\vec{l} = \rho\, d\tau\, \vec{E}\cdot d\vec{l}$$

It's useful to consider rate of doing work so suppose charges move $d\vec{l}$ in dt:

$$\frac{dW_{part}}{dt} = \rho \vec{E}\cdot \frac{d\vec{l}}{dt}\, d\tau$$

$$= \boxed{\rho \vec{E}\cdot\vec{v}}\, d\tau$$

But the current density $\vec{J} = \rho\vec{v}$

$$\frac{dW_{part}}{dt} = \vec{J}\cdot\vec{E}\, d\tau = \frac{\partial W_{part}}{\partial t}\, d\tau$$

If we integrate over all charges in system

For the entire system

$$\frac{dW_{part}}{dt} = \int d\tau\, \vec{J} \cdot \vec{E}$$

Note Griffiths uses u for density
W for integral over volume

So we've got particles undergoing forces and accelerating, increasing their energy. Where does this energy come from? Energy conservation: Must come from fields $\vec{E}, \vec{B}$

The energy density of the fields is given by

electric $u_e = \frac{1}{2} \epsilon_0 E^2$
magnetic $u_m = \frac{1}{2} \mu_0 B^2$

Total energy $W = \int d\tau\, (u_{e\, elem} + u_{part})$

The energy that allows charges to accelerate must come from the fields

Let's look for an expression involving energy conservation

Take $\frac{\partial u_e}{\partial t} + \frac{\partial u_m}{\partial t} = \vec{E} \cdot \epsilon_0 \frac{\partial \vec{E}}{\partial t} + \frac{1}{\mu_0} \vec{B} \cdot (\vec{\nabla} \times \vec{E})$

Use Maxwell to relate to forces on charges

Use Maxwell

$$= \vec{E}\cdot\left(\frac{\vec{\nabla}\times\vec{B}}{\mu_0} - \vec{J}\right) + \frac{1}{\mu_0}\vec{B}\,(-)(\vec{\nabla}\times\vec{E})$$

$$= \frac{1}{\mu_0}\left(\vec{E}\cdot(\vec{\nabla}\times\vec{B}) - \vec{B}\cdot(\vec{\nabla}\times\vec{E})\right) - \vec{E}\cdot\vec{J}$$

[See next page for $\nabla\cdot(\vec{E}\times\vec{B})$]

so $$\frac{\partial u_e}{\partial t} + \frac{\partial u_m}{\partial t} = \underbrace{-\vec{E}\cdot\vec{J}}_{\frac{\partial u_{part}}{\partial t}} \cancel{+} \frac{\vec{\nabla}\cdot(\vec{E}\times\vec{B})}{\mu_0}$$

$$\vec{S} \equiv \frac{\vec{E}\times\vec{B}}{\mu_0}$$ Poynting vector

so we have $$\frac{\partial u_e}{\partial t} \cancel{+} \frac{\partial u_m}{\partial t} + \frac{\partial u_{part}}{\partial t} = \vec{\nabla}\cdot\vec{S}$$

$\vec{S}$ gives the direction of the time rate of change of energy

$$[S] = \left(\frac{[E][B]}{\mu_0}\right) = \left[\frac{F}{q}\right]\frac{I}{l} = \frac{F\,l}{t\,l^2} = \frac{\text{Work}}{\text{Area time}}$$ = Energy

$\vec{S}$ is the Energy flux density of e & m fields

defining $u = u_e + u_m + u_{part}$ we see that

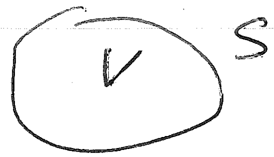
Poynting's thm $$\boxed{\vec{\nabla}\cdot\vec{S} \cancel{+} \frac{\partial u}{\partial t} = 0}$$

expression of energy conservation

analogous to $\vec{\nabla}\cdot\vec{J} \cancel{+} \frac{\partial \rho}{\partial t} = 0$ charge conservation

Take a volume V integrate over it

V fixed in space



$$\frac{dW}{dt} = -\int d^3r\, \vec{\nabla}\cdot\vec{S} = -\oint_S d\vec{a}\cdot\vec{S}$$

Δ energy inside $= -\Delta$ energy outside   energy conservation

$$\vec{\nabla}\cdot(\vec{E}\times\vec{B})$$

$$= \frac{\partial}{\partial x_n} E_i B_j \epsilon_{ijn}$$

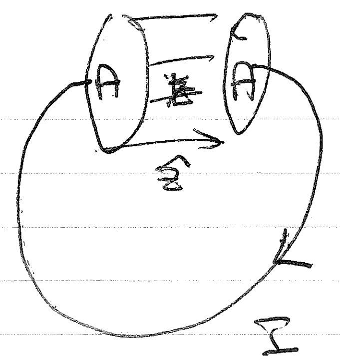
$$= \left[+\frac{\partial E_i}{\partial x_n}\epsilon_{inj}\right] B_j \pm E_i\left(\frac{\partial B_j}{\partial x_n}\epsilon_{ijn}\right)$$

jth component of $\vec{\nabla}\times\vec{E}$

ith component of $\vec{\nabla}\times B$

So $\vec{\nabla}\cdot(\vec{E}\times\vec{B}) = +\vec{B}\cdot(\vec{\nabla}\times\vec{E}) - \vec{E}\cdot(\vec{\nabla}\times\vec{B})$

Example Charging capacitor



assume charge on plates

examine region between plates

Goal compute ∇B, $\vec S$ verify

$$-\nabla\cdot\vec S = \frac{+\partial(u_e+u_m)}{\partial t}$$

Compute $\vec E$, $\vec B$

$$\vec E = \frac{\sigma}{\epsilon_0}\hat k \qquad \sigma(t) = \frac{Q}{A} = \frac{It}{A},$$

$$\vec\nabla\times\vec B = +\mu_0\epsilon_0\frac{\partial\vec E}{\partial t} = +\mu_0\frac{I}{A}\hat k$$

$\vec B$ in $\hat\phi$ direction integrate over area inside circle of radius s

$$\int(\nabla\times\vec B)\cdot d\vec a = \int d\vec l\cdot\vec B = 2\pi s B = +\mu_0\frac{I}{A}\pi s^2$$

$$\vec B = +\mu_0\frac{I}{A}\frac{s}{2}\hat\phi$$

$$u_e+u_m = \frac{1}{2}\frac{I^2t^2}{\epsilon_0 A^2} + \frac{1}{2}\mu_0\frac{I^3}{A^2}\frac{s^3}{4}$$

$$= \frac{1}{2}\frac{I^3}{A^2}\left[\frac{t^3}{\epsilon_0} + \frac{\mu_0 s^2}{4}\right]$$

$$\vec S = \frac{\vec E\times\vec B}{\mu_0} = -\frac{1}{\mu_0}\frac{It}{A}\frac{\mu_0 I}{A}\frac{s}{2}\hat s$$

$$= -\left(\frac{I}{A}\right)^2\frac{t}{2}\vec s$$

$$\frac{\partial (u_e + u_m)}{\partial t} = \frac{I^2}{A^2}\,\frac{t}{\epsilon_0}$$

$$\vec{\nabla}\cdot\vec{S} = -\left(\frac{I^2}{A^2}\right) t \qquad \text{using } \vec{\nabla}\cdot\vec{S} \to \frac{1}{s}\frac{\partial}{\partial s}(s\, S_s)$$

So Poynting thm verified

for region between plates

$$\vec{\nabla}\cdot\vec{S} + \frac{\partial (u_e + u_m)}{\partial t} = 0$$

Local
Conservation of energy is done

Now Want to do same for momentum
need mathematical detour

Tensor, Vector

p10, p11, 12
Griffiths

What is a vector?

magnitude direction

what does this mean?

I have a height and I am walking

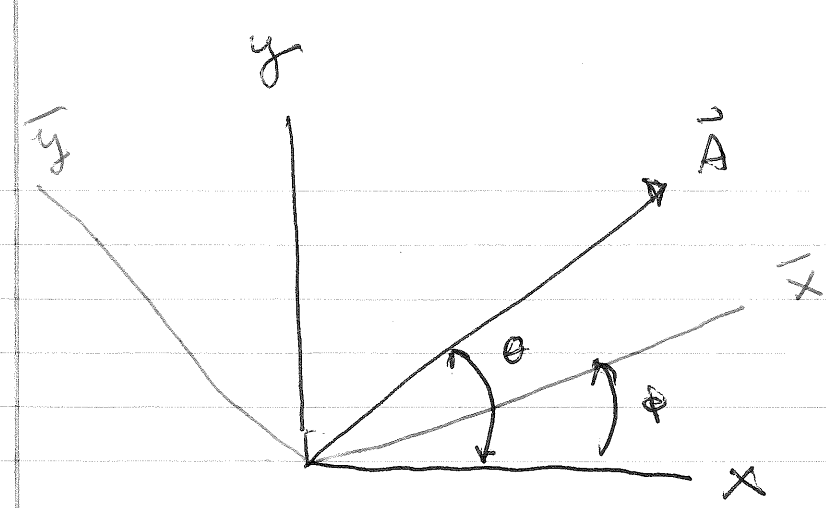
Am I a vector?

Vector has 3 components

Barrel of beer with Bud Rainier and Michelob

Is that a vector 3 components
can add

Vector transform specific way
when coordinate axes are rotated
Consider vector $\vec{A}$ and 2 axes
two dimensions



$A_x = A\cos\theta$, $A_y = A\sin\theta$

$$A_{\bar{x}} = A\cos(\theta - \phi)$$
$$= A\cos\theta\cos\phi + A\sin\theta\sin\phi$$
$$= A_x\cos\phi + A_y\sin\phi$$
$$A_{\bar{x}} = A_x\cos\phi + \sin\phi\, A_y$$

$$A_{\bar{y}} = A\sin(\theta - \phi)$$
$$= A\sin\theta\cos\phi - A\cos\theta\sin\phi$$
$$= \cos\phi\, A_y - \sin\phi\, A_x$$

. Compact notation

$$\begin{pmatrix} A_{\bar{x}} \\ A_{\bar{y}} \end{pmatrix} = \begin{pmatrix} \cos\phi & \sin\phi \\ -\sin\phi & \cos\phi \end{pmatrix} \begin{pmatrix} A_x \\ A_y \end{pmatrix}, \quad \bar{A} \equiv (A_{\bar{x}}, A_{\bar{y}})$$

Compact notation $\bar{A} = RA$

$\bar{A}_i = R_{ij} A_j$ sum on repeated indices

$A_x = R_{xx}A_x + R_{xy}A_y + R_{xz}A_z$
$= R_{11}A_1 + R_{12}A_2 + R_{13}A_3$

Can do this for 3 indices $x\,y\,z$ or $1\,2\,3$

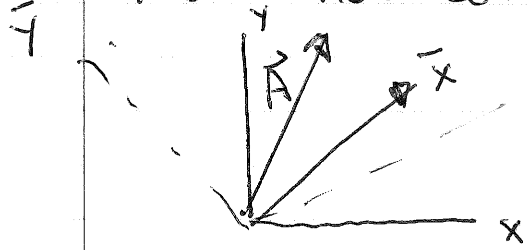
This shows how coordinates of a displacement transforms

A set of 3 components that transforms

$$\begin{pmatrix} \bar{A}_{\bar{x}} \\ A_{\bar{y}} \end{pmatrix} = \begin{pmatrix} \cos\phi & \sin\phi \\ -\sin\phi & \cos\phi \end{pmatrix} \begin{pmatrix} A_x \\ A_y \end{pmatrix}$$

Last time introduced vector definition

it's how components transform



$$\bar{A}_i = \sum_j R_{ij} A_j$$

R_{ij} = component 2×2 matrix last time

more general 3×3

other notation $\bar{A}_i = R_{ij} A_j$ sum over repeated twice indices

or even $\bar{A} = RA$

A displacement vector transforms the same way A set of 3 components that transforms

Same as displacements is a vector

$\vec{p} = m d\vec{r}/dt$, $\vec{F} = m d^2\vec{r}/\cancel{dt^2} = \frac{d\vec{p}}{dt}$ etc

Length is preserved under rotation of axes
must not depend on axes choice

~~Length~~ Displacement $= \begin{bmatrix} x \\ y \end{bmatrix} = V$

Length^2 $= V^T V$ $\qquad \bar{A}^T = A^T R^T$

$= [x\ y] \begin{bmatrix} x \\ y \end{bmatrix} = x^2 + y^2$

$$\bar{A}^T \bar{A} = [A_x\ A_y] \left(\begin{bmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{bmatrix} \begin{bmatrix} \cos\phi & \sin\phi \\ -\sin\phi & \cos\phi \end{bmatrix} \right) \begin{pmatrix} A_x \\ A_y \end{pmatrix}$$

$$\downarrow$$

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$\bar{A}^T \bar{A} = A^T A = A_x^2 + A_y^2$

More general $\bar{A}^T A^T = A^T (R^T R) A$

$R^T R = \underline{I}$

For rotations: the transpose of a rotation matrix is its inverse

We can also define a two- rank

tensor

T_{ij} for example $x_i x_j$

$$\bar{T}_{ij} = R_{in} R_{jm} T_{nm}$$

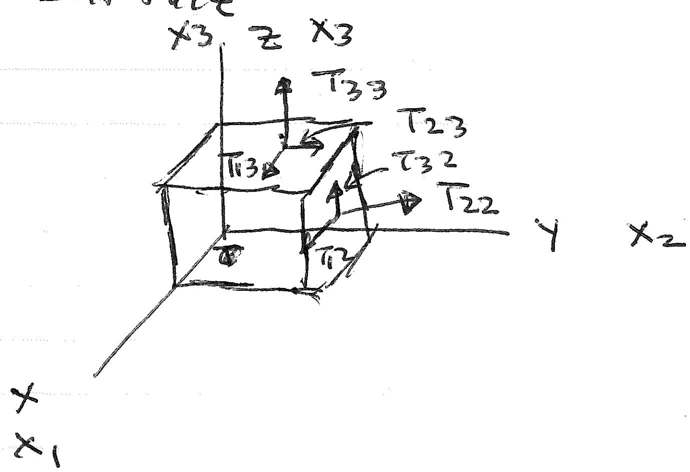
moment of inertia tensor

Example Force per unit area in 3D space needs nine components
3 for force and X 3 for normal to area

Describe force per unit area as tensor
T_{ij} the first index specifies

stress tensor

direction in which the stress (force) acts
the second index identifies direction normal to surface



x, y, z
$= 1, 2, 3$

We need this math to describe momentum conservation

with this notation $\vec{p}, \frac{d\vec{p}}{dt}, \vec{r}$
a vector is a rank 1 tensor

We can make a dot product

between a vector and a rank two tensor

$V_j = r_i T_{ij}$ now has 3 components

transforms as a vector. Easy way same as leaves V_i

Proof

$\bar{r}_i = R_{in} r_n$

$\bar{T}_{ij} = R_{ia} R_{jb} T_{ab}$

$\bar{r}_i \bar{T}_{ij} = (R_{in} R_{ia}) R_{jb} r_n T_{ab}$

R^T_{ni}, $R^T_{ni} R_{ia}$ = matrix product of inverse times matrix

$= \delta_{na} \quad R_{jb} r_n T_{ab}$

$= R_{jb} (r_n T_{nb}) = R_j$

R_{in} is inverse of R_{ni} $= R^T$

$r_i T_{ij}$ is written in compact notation

$T_{ij} = \overleftrightarrow{T}$

$(\vec{r} \cdot \overleftrightarrow{T}) =$ Vector

$(\vec{r} \cdot \overleftrightarrow{T})_j = r_i T_{ij}$

Another example $T^E_{ij} = E_i E_j$

$\vec{\nabla} \cdot \overleftrightarrow{T} = \frac{\partial E_i}{\partial x_i} E_j$

here defined to act on first $\vec{E}$ only

This math is meant to understand

Momentum of E&M Field

E&M field carries momentum as well as energy. You have heard about this in 122.

Try to explain in terms of photons

wave particle duality

E&M waves collection of photons γ

$E_\gamma = p_\gamma c$

Light beam going in a given direction

Density of photons $= n_\gamma$

each has E_γ, $\vec{p}_\gamma$

Energy density $u = n_\gamma E_\gamma$

(energy flux density) $\overset{= uc}{}$ $S = E_\gamma n_\gamma c$ poynting vector

momentum density $\vec{\wp} = n_\gamma \vec{p}_\gamma = \left(\frac{n_\gamma E_\gamma}{c}\right) = \frac{S}{c^2}$

momentum density

$$\vec{\wp} = \frac{1}{c^2}\vec{S} = \mu_0 \epsilon_0 \vec{S}$$

$\vec{\wp} = \frac{\vec{S}}{c^2}$

Will ~~not~~ give derivation of Mom. conservation but more important than derivation is the idea that E&M fields carry momentum

$\vec{S} = \vec{E} \times \vec{B}$

$\vec{S}/c^2$ = momentum density - fields carry momentum

Basic idea of momentum conservation

Fields carry energy

$\Rightarrow$ Fields carry momentum

Energy & momentum are components of a four vector

E & B fields made of vast numbers of photons $\leftarrow$ particles of 0 mass

$$E = \sqrt{m^2c^4 + p^2c^2}$$

if $m=0$ $\quad E = pc$

$$p = \frac{E}{c}$$

Expect a relation like this for $\vec{E}$ & $\vec{B}$

Should involve $\vec{E}, \vec{B}$ & S

want conservation law like $\vec{\nabla}\cdot\vec{J} + \frac{\partial \rho}{\partial t} = 0$ $\quad \vec{\nabla}\cdot\vec{S} + \frac{\partial u}{\partial t} = 0$

Basic idea consider force on charged particle $\quad \vec{F} = \frac{d\vec{p}}{dt}$ (analogous to $\frac{\partial u}{\partial t}$)

$dq = \rho\, d\tau$ $\qquad \vec{F} = \int_V (\rho\, d\tau)(\vec{E} + \vec{J}\times\vec{B})$

force density $\vec{f}$ relate to $\frac{d}{dt}$ (momentum density)

$$\vec{f} = \rho\vec{E} + \vec{J}\times\vec{B} = \frac{\partial \vec{p}_{mech}}{\partial t}$$

This depends on ρ, J $\quad$ - rewrite in favor of fields

$$\rho = \epsilon_0 \vec{\nabla}\cdot\vec{E}, \quad \vec{J} = \frac{1}{\mu_0}(\vec{\nabla}\times\vec{B}) - \epsilon_0\frac{\partial \vec{E}}{\partial t}$$

$$\vec{f} = \epsilon_0(\vec{\nabla}\cdot\vec{E})\vec{E} + \frac{1}{\mu_0}(\vec{\nabla}\times\vec{B})\times\vec{B} - \epsilon_0\left(\frac{\partial\vec{E}}{\partial t}\times\vec{B}\right)$$

almost Poynting vector

$$\frac{\partial\vec{E}}{\partial t}\times\vec{B} = \frac{\partial}{\partial t}(\vec{E}\times\vec{B}) + \vec{E}\times\frac{\partial\vec{B}}{\partial t}$$

$$= \frac{\partial}{\partial t}(\vec{E}\times\vec{B}) - \vec{E}\times(\vec{\nabla}\times\vec{E}))$$

looks like

Collect terms

$$\vec{f} = \epsilon_0\left[(\vec{\nabla}\cdot\vec{E})\vec{E} - \vec{E}\times(\vec{\nabla}\times\vec{E})\right]$$

$$+ \frac{1}{\mu_0}\left[\nabla\cdot B\,\vec{B} - \vec{B}\times(\vec{\nabla}\times\vec{B})\right] - \epsilon_0\frac{\partial}{\partial t}(\vec{E}\times\vec{B})$$

$\nabla\cdot B \to 0$ ok to add

We have quadratic terms in $\vec{E}$ & $\vec{B}$
expect energy density to enter
Need vector analysis

$$\vec{E}\times(\vec{\nabla}\times\vec{E}) = \frac{1}{2}\vec{\nabla}E^2 - (\vec{E}\cdot\vec{\nabla})\vec{E}$$

same for $\vec{B}$ so

$$\vec{f} = \epsilon_0\left[(\vec{\nabla}\cdot\vec{E})\vec{E} + (\vec{E}\cdot\vec{\nabla})\vec{E}\right]$$

$$+ \frac{1}{\mu_0}\left[(\vec{\nabla}\cdot\vec{B})\vec{B} + (\vec{B}\cdot\vec{\nabla})\vec{B}\right]$$

$$- \vec{\nabla}\left(\frac{\epsilon_0}{2}E^2 + \frac{B^2}{2\mu_0}\right) - \epsilon_0\frac{\partial(\vec{E}\times\vec{B})}{\partial t}$$

Aside proof that

$$\vec{E} \times (\vec{\nabla} \times \vec{E}) = \frac{1}{2} \nabla E^2 - (\vec{E} \cdot \vec{\nabla}) \vec{E}$$

$$(\text{LHS})_\alpha = E_i \frac{\partial}{\partial x_j} E_k \quad \epsilon_{jkn} \epsilon_{in\alpha} \qquad \stackrel{?}{=} \text{RHS}_\alpha$$

$$\epsilon_{jkn} \epsilon_{in\alpha} = \epsilon_{jkn} \epsilon_{\alpha i n}$$

$$= \delta_{j\alpha} \delta_{ki} - \delta_{ji} \delta_{k\alpha}$$

$$(\text{RHS})_\alpha = E_i \frac{\partial}{\partial x_\alpha} E_i - E_i \frac{\partial}{\partial x_i} E_\alpha$$

$$= \frac{1}{2} \frac{\partial}{\partial x_\alpha} E_i E_i - (\vec{E} \cdot \vec{\nabla}) E_\alpha$$

So $\overrightarrow{\text{LHS}} = \frac{1}{2} \vec{\nabla} \hat{E}^2 - (\vec{E} \cdot \vec{\nabla}) \vec{E}$

The terms involving E & B

Look like a gradient of a tensor

$x\,y\,z = 1,2,3$

$$T_{ij} \equiv \epsilon_0 \left(E_i E_j - \frac{1}{2}\delta_{ij} E^2\right) + \frac{1}{\mu_0}\left(B_i B_j - \frac{1}{2}\delta_{ij} B^2\right) = \text{Maxwell Stress Tensor}$$

- Let's check this

$\frac{\partial}{\partial x_i} T_{ij} =$ is jth component of a vector

lets see just look at E terms

$$\frac{\partial}{\partial x_i} T_{ij} = \epsilon_0 \left(\frac{\partial E_i}{\partial x_i} E_j + E_i \frac{\partial E_j}{\partial x_i} - \frac{1}{2}\delta_{ij}\frac{\partial}{\partial x_i} E^2 \right)$$

$$= \epsilon_0 \left(\vec{\nabla}\cdot\vec{E}\, E_j + \vec{E}\cdot\vec{\nabla} E_j - \frac{1}{2}\nabla_j E^2 \right)$$

this what we had before we see that B terms are the same

This means that

$$f_j = \frac{\partial}{\partial x_i} T_{ij} - \underbrace{\epsilon_0\mu_0}_{= 1/c^2} \frac{\partial \vec{S}}{\partial t}$$

In vector notation

- $$\vec{f} = \vec{\nabla}\cdot\overleftrightarrow{T} - \frac{1}{c^2}\frac{\partial \vec{S}}{\partial t}$$

$$\vec{f} = \frac{\partial}{\partial t}\vec{P}_{mech}, \quad \vec{P}_{em} = \frac{\vec{S}}{c^2}$$

So we have

$$\underbrace{\frac{\partial \vec{P}_{mech}}{\partial t}}_{\text{Force exerted by particles \& fields}} + \frac{\partial \vec{P}_{em}}{\partial t} = \vec{\nabla}\cdot\overleftrightarrow{T}$$ momentum

Integrate over ^fixed Volume $\mathcal{V}$ with surface $\mathcal{S}$

$$\frac{d\vec{P}_{mech}}{dt}{}_{field} = \int_{\mathcal{V}} d\tau \frac{\partial P_{mech}}{\partial t}{}_{field} \quad \text{et.}$$

Take component

$$\boxed{\frac{d}{dt}\left(\vec{P}_{mech} + \vec{P}_{field}\right)_i = \int_{\mathcal{V}} d\tau \left(\vec{\nabla}\cdot\overleftrightarrow{T}\right)_i = \oint_{\mathcal{S}} da_n T_{ni}}$$

Use divergence theorem

$$\vec{\nabla}\cdot\overleftrightarrow{T} = \frac{\partial}{\partial x_n} T_{ni}$$

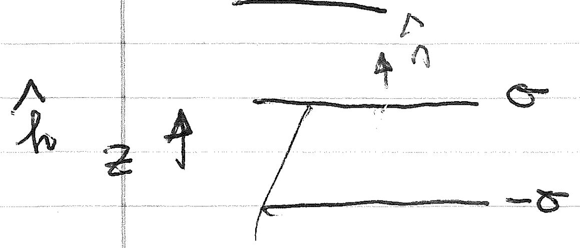
This is conservation of momentum

Change of momentum ^wrt ^time inside $\mathcal{V}$ is equal to change of momentum through surface. This equation (boxed) can be used to calculate forces on material objects by computing flux

The force transmitted across $\mathcal{S}$ acting on combined system of particles and fields inside $\mathcal{V}$

Example

Parallel plate cap



a) determine T_{ij}

b) Force on top plate ← can get old way

c) EOM momentum per unit area crossing plate per unit time

write T_{ij} as 3x3 matrix

a) $\vec{E} = -\frac{\sigma}{\epsilon_0}\hat{k}$ $\qquad E_i = -\frac{\sigma}{\epsilon_0}\delta_{i3} = -E_0\delta_{i3}$ $\qquad E_0 > 0$

$T_{ij} = \epsilon_0\left(E_iE_j - \frac{\delta_{ij}}{2}\vec{E}^2\right)$ $\qquad \vec{B} = 0$

if $i \neq j$ $\quad T_{ij} = 0$

$T_{33} = +\frac{\epsilon_0}{2}E_0^2 = \frac{\sigma^2}{2\epsilon_0}$

$T_{11} = -\frac{\epsilon_0}{2}E_0^2 = T_{22}$

b) $\vec{F}_0 = \int d\vec{a} \cdot \overleftrightarrow{T}$ $\qquad d\vec{a} = +da\,\hat{k}$ outward to surface enclosing plate

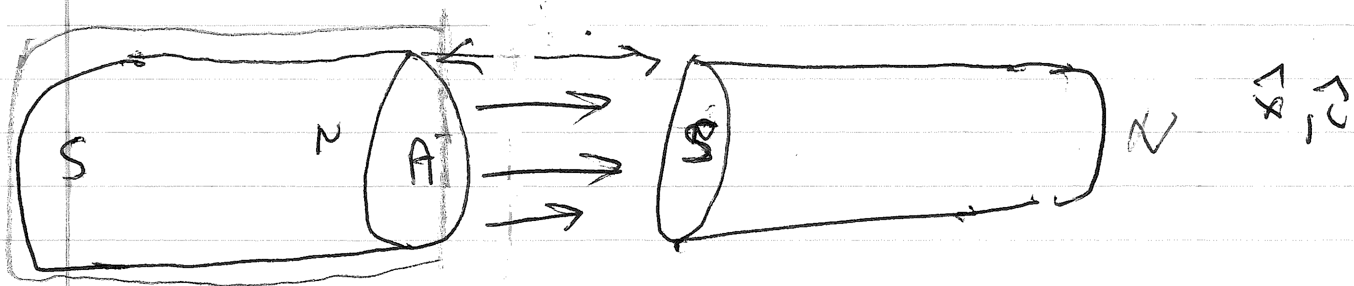
$F_i = +\int da\, T_{i3}$ $\qquad F_{1,2} = 0$ Makes sense

$F_3 = +A\,T_{33} = -\frac{\sigma^2}{2\epsilon_0}A$

c) Moment/area/time = Force/area $= +T_{33} = \sigma^2/2\epsilon_0$

Another example

Between Poles of Magnet



Consider attraction between 2 bar magnet

Concentrate on region between poles, largest fields. Take $\vec{B}$ to be uniform
to complete T_{ij}

$\vec{B} = B\hat{x}$ $\quad$ $B=0$ in other regions

Lets use Maxwell Stress tensor to get the force on the bar magnet

a) What is T_{ij}?

$$T_{11} = \frac{B^2}{2\mu_0}$$

$$T_{22} = -\frac{B^2}{2\mu_0}$$

$$T_{33} = -B^2/2\mu_0$$

$\frac{B^2}{2\mu_0}$ is energy density u

so $$T = u\begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

b) Now compute force on the left magnet

$$\vec{f} = \vec{\nabla}\cdot\overleftrightarrow{T} - \frac{1}{c^2}\frac{\partial \vec{S}}{\partial t} \qquad \vec{S} = 0$$

The x component is the interesting one

$$f_x = \vec{\nabla}\cdot\overleftrightarrow{T}_x$$

$$= \frac{\partial}{\partial x_n} T_{nx} = \frac{\partial}{\partial x} T_{xx} + \frac{\partial}{\partial y} T_{yx} + \frac{\partial}{\partial z} T_{zx}$$

integrate over volume indicated by $\square$

$$= \frac{\partial}{\partial x} T_{xx} = \frac{\partial T_{11}}{\partial x_1} + \frac{\partial}{\partial x_2} T_{21} + \frac{\partial T_{31}}{\partial x_3}$$

$$\int d^3r \, f_x = F_x = \int d^3r \frac{\partial}{\partial x} T_{xx}$$

$$d^3r = A\,dx$$

So $$F_x = A \int_{\text{Left}}^{\text{Right}} dx \frac{\partial T_{xx}}{\partial x}$$

$$= A\, T_{xx}(\text{Right})$$

$$= A\,\mathcal{U} \qquad \text{to right}$$

c) Energy in the field

$$W = \mathcal{U}\ \text{Volume} = \mathcal{U} A L$$

Electromagnetic fields carry energy and momentum and angular momentum

Poynting Vector

$$\vec{S} = \frac{1}{\mu_0} (\vec{E} \times \vec{B})$$ energy flux

$\frac{\text{energy}}{\text{Area time}}$

$\vec{P}_{em}$ is momentum density : momentum per unit volume carried by $\vec{E}$ & $\vec{B}$

$$= \epsilon_0 (\vec{E} \times \vec{B}) = \frac{\vec{S}}{c^2}$$

7 One more item worth mentioning now we understood intuitively the relation between $\vec{P}_{em}$ & $\vec{S}$ from the notion of photons and the relation between energy & momentum of photon. Photons carry angular momentum

The angular momentum density

$$\mathcal{L}_{em} = \vec{l} = \vec{r} \times \vec{P}_{em}$$

There is something amazing or puzzling here. As long as there is $\vec{E}$ & $\vec{B} \neq 0$

$\vec{S} \neq 0$

and $\vec{E}$ is not parallel to $\vec{B}$ there

is energy flow, momentum and angular

momentum. How can this be? The point

is that the conservation of energy momentum

and angular momentum will not work without

the terms related to Poynting vector.

We've discussed the conservation laws for

energy starting with work done by fields on charges
momentum " " forces by fields on charge

Similarly for angular momentum we could

consider $\vec{f}$ force density $= \rho\vec{E} + \vec{J} \times \vec{B}$

and $\vec{r} \times \vec{f}$ and get an conservation eqn

energy: $\frac{\partial u}{\partial t} = -\vec{\nabla} \cdot \vec{S}$ $u =$ electromagn energy density

$\frac{\epsilon_0 E^2}{2} + \frac{B^2}{2\mu_0}$

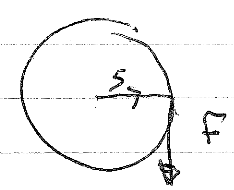
Momentum $\vec{f} = \vec{\nabla} \cdot \overleftrightarrow{T} - \frac{1}{c^2}\frac{\partial \vec{S}}{\partial t}$

can get a similar eqn for angular momentum

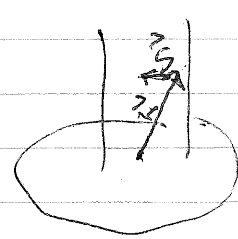
$$N = \vec{r} \times \vec{F}$$

$$= \vec{r} \times \frac{d\vec{p}}{dt} = \frac{d}{dt}(\vec{r} \times m)$$

$$= \frac{d}{dt}\vec{L}$$

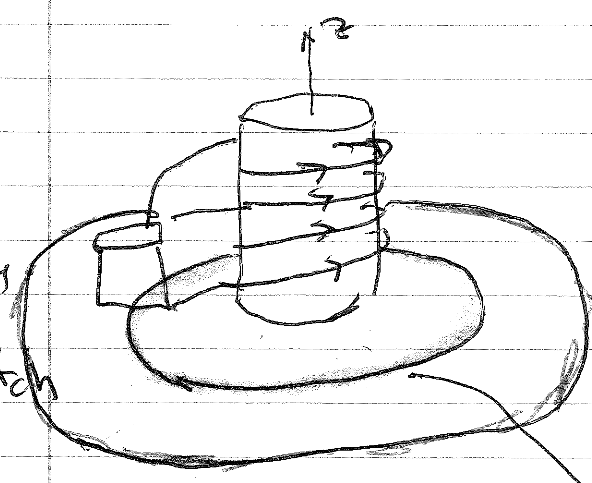


rotates clockwise



Angular momentum

Example Feynman's disk paradox



insulator disk free to rotate around its axis like a spool of thread

attach a solenoidal coil

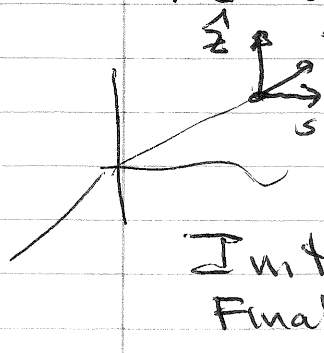
Initially battery not connected to coil, no current flows system is at rest

Now connect battery to solenoid by closing switch, current flows in direction shown

this cause $\vec{B}$ in the z direction, While changing time dependent B causes Emf produced in $-\hat{\phi}$ direction There is an $\vec{E}$ in the $-\hat{\phi}$ direction This cause a force and a torque on the ring of charge

The torque is in the direction of $\vec{s} \times \vec{f}$

$$= \vec{s} \times -\hat{\phi} \propto -\hat{z}$$



disk rotates clockwise

Initial angular momentum was 0

Final ang mom should be 0??

Resolution of apperent paradox

The electromagnetic field carries

ang mom. density $\vec{r} \times \dfrac{\vec{S}}{c^2} = \vec{r} \times (\vec{E} \times \vec{B})$

This is equal and opposite to mechanical angular momentum gained by disk

get direction
$\vec{E}$ in $-\hat{\phi}$
$\vec{B}$ in $\hat{z}$ inside coil

$\vec{E} \times \vec{B} =$ in $\hat{z} \times \hat{\phi} = -\vec{s}$ direction

$\vec{r} \times (\vec{E} \times \vec{B}) = z\hat{h} \times (-\hat{s})$

$= z\, \hat{s} \times \hat{h}$ $+\hat{\phi}$ direction

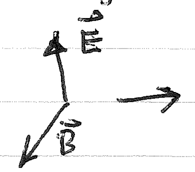
Will skip section 8.4 as it is already covered

Will go to waves -

Wave nature of light

We will see the true meaning of the Poynting vector when we study this subject. Maxwell eqns teach us that light is an electromagnetic wave

looking ahead



direction of propagation

Light is a transverse electromagnetic wave

But we need to review general properties of waves. May have been a long time since 123 or its equivalent

---

Demos Tension dependence of wave speed 3B10.15

waves plucked on stretched rubber shows Bell Labs Machine

dependence on tension

Spring on Table 3B10.26

We want a mathematical description

Basic definition of a wave

Wave is a disturbance from equilibrium that travels (propagates) from one region of space to another. ~~that~~ It carries energy not matter.

Sound waves carry energy and information thru the air (air doesn't move)

Ocean waves bring to shore energy imparted by distant winds and other forces

Earthquakes setup wave motion in the earth itself

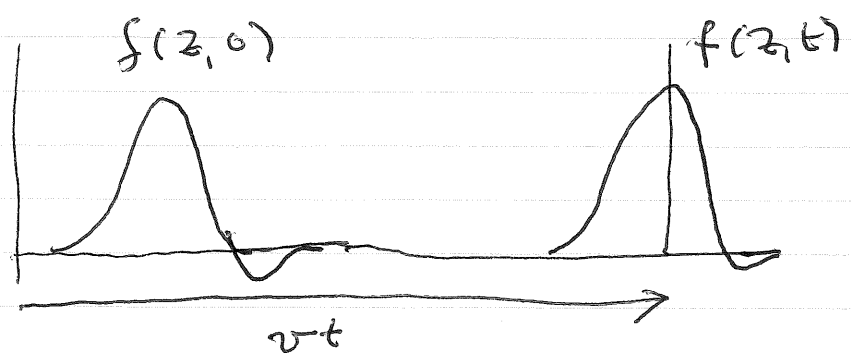
Ultrasound waves probe human body

The above are mechanical disturbances easy to visualize. Our focus will be on EM waves, but we'll begin at beginning

Consider a traveling disturbance that travels with fixed shape at constant velocity.

The mathematical representation is of amplitude magnitude of disturbance

one functio



$v > 0$

This function captures the essence of

~~The special thing about~~ waves:

$$f(z,t) = g(z-vt) \quad \text{or} \quad g(z+vt) \qquad v>0$$

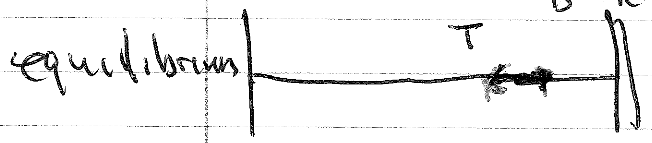
g can be anything

$$g(z-vt) = A e^{-b(z\pm vt)^2}$$

$$\text{or } A \cos b(z\mp vt)$$

Example 1 - string under tens

is tension

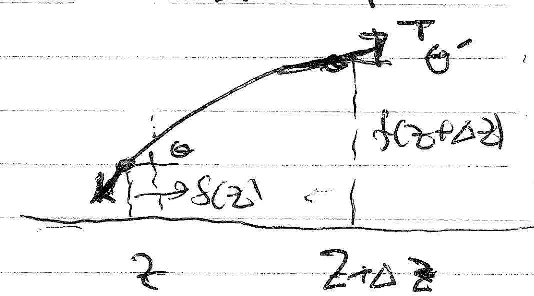


small deviation from horizontal due to gravity

string has mass per unit length $= \mu$

Then pluck string by <u>small</u> vertical displacement

Look at blow up



T acts along string

each piece feels left + right force

f is displacement from horizontal

goal calculate transverse force on string

$$\Delta F = T(\tan\theta' - \tan\theta)$$

$$= T\left(\frac{\partial f}{\partial z}(z+\Delta z) - \frac{\partial f}{\partial z}(z)\right)$$

$$= T \Delta z \frac{\partial^2 f}{\partial z^2}$$

Now use ~~F~~ Newton's second eqn
$F = ma$

$$\Delta F = \mu \Delta z \frac{\partial^2 f}{\partial t^2}$$

why small displacements - ignore horizontal effects

so we have
$$\frac{\partial^2 f}{\partial z^2} = \frac{\mu}{T} \frac{\partial^2 f}{\partial t^2}$$

analyze this equation

$$\left[\frac{\mu}{T}\right] = \left[\frac{m / \text{length}}{\frac{m \text{ length}}{t^2}}\right] = \frac{t^2}{l^2} = \frac{1}{v^2}$$

$$\frac{\partial^2 f}{\partial z^2} = \frac{1}{v^2} \frac{\partial^2 f}{\partial t^2}$$

solution $f(z,t) = g(z \pm vt)$

where g is any 2nd differentiable function

check let $u = z \pm vt$

$$\frac{\partial f}{\partial z} = \frac{dg}{du} \frac{\partial u}{\partial z} = \frac{dg}{du}$$

$$\frac{\partial^2 f}{\partial z^2} = \frac{\partial^2 g}{\partial u^2} \frac{\partial u}{\partial z} = \frac{\partial^2 g}{\partial u^2} = g''$$

$$\frac{\partial f}{\partial t} = \frac{\partial g}{\partial u} \frac{\partial u}{\partial t} = \pm v \frac{\partial g}{\partial u} = \pm v g'$$

$$\frac{\partial^2 f}{\partial t^2} = \pm v \frac{\partial^2 g}{\partial u^2} \frac{\partial u}{\partial t} = v^2 \frac{\partial^2 g}{\partial u^2} = v^2 g''$$

g can be any thing

Speccal form of waves
Sinusoidal

$f(z,t) = A \cos (k(z-vt)+\delta)$

is Most familiar because of

Fourier's theorem - any function can

be written is linear combination of cos' or

sines

A = amplitude = maximum displacement

phase $k(z-vt)$

phase constant $= \delta$

k = wave number

$= \frac{2\pi}{\lambda}$ = wave length

for fixed t. if z increases by

λ the wave repeats itself argument increases by 2π

T = period $= \frac{2\pi}{kv} = \frac{\lambda}{v}$ if z is fixed

and t increased by T ~~fixed~~

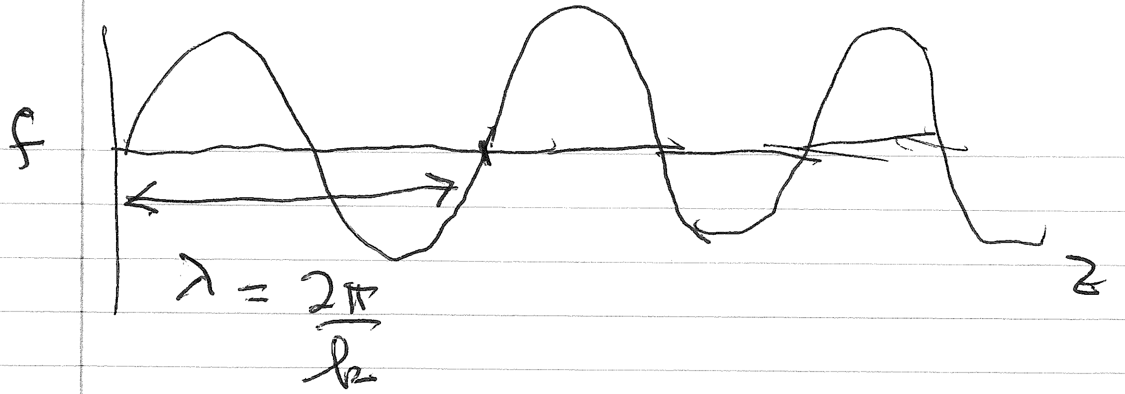
argument increases by 2π & function

repeats $\nu = \frac{1}{T}$ = frequency [Hz] (s^{-1})

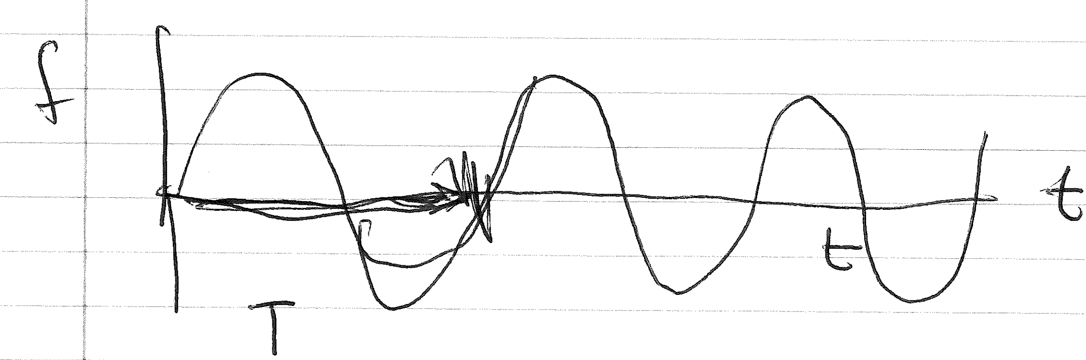
ang frequency $\omega = 2\pi\nu$

$f(z,t) = A \cos (kz - \omega t + \delta)$

fixed t



fixed z



Physics is not a spectator sport

Surfer paddles out to deep water region where ocean waves are sinusoidal in shape with crests 16 m apart

surfer rises a vertical distance of 4 m from wave trough to crest, which takes 2 s.
Crest is at $z=0$ at $t=0$
write $f(z,t)$

Time from trough to crest is $T/2$ so $T = 8$ s

$\lambda = 16$ m ~~$2a$~~ $v = \frac{\lambda}{T} = \frac{16\text{ m}}{2\text{ s}} = 8$ m/s

The amplitude $A = \frac{1}{2}$ vertical distance $= 2$ m

$$f(z,t) = A\cos(k(z - vt))$$

$$= 2m \cos\left(\frac{2\pi}{16m}\left(z - \frac{8m}{s}t\right)\right)$$

Now there is a better way to represent wave

Using Euler's eqn $e^{i\theta} = \cos\theta + i\sin\theta$

$$f(z,t) = \mathrm{Re}\, A e^{i(kz - wt + \delta)}$$

$$\text{Or } \mathrm{Re}\left(\left(Ae^{i\delta}\right) e^{i(kz-wt)}\right)$$

$$= \mathrm{Re}\, \tilde{A} e^{i(kz-wt)}$$

In general wave ~~moving in +z direction can~~

can be written

$$f(z,t) = \int_{-\infty}^{\infty} dk\, \tilde{A}(k) e^{i(kz-wt)}$$

Go back to string theory

I want to show that there is a flow of energy down the spring. Can compute rate of flow of energy = Power

Compute power delivered from $\rightarrow$

left to right at point z

= rate of doing work (force x distance)

Power = P = upward velocity at point z

x upward force exerted by left on right

Velocity is $\frac{\partial f}{\partial t}$

Force $T\frac{\partial f}{\partial z}$ ∂

Product $P = T\frac{\partial f}{\partial z}\frac{\partial f}{\partial t}$

write in terms of g

$$\frac{\partial f}{\partial z} = k g'$$

$$\frac{\partial f}{\partial t} = kv\, g' = \omega g'$$

$$\underline{P} = k\omega T \; (g'^2) > 0$$

$$g'^2 \sim A^2 \cos^2(kz - \omega t)$$

So for light ω is very large take time average over 1 cycle: def is $\frac{1}{T}\int_0^T dt\,(P)$

$$\bar{P} = \frac{1}{2}A^2 k\omega T$$

Maxwell's eq - partial diff eqns

to solve need BC

needed for interaction between light & matter

Boundary Conditions

1

2

In general E, B, D, H not continuous at surface that carries σ or $\vec{K}$

$$\vec{\nabla}\cdot\vec{D} = \rho_f \rightarrow \oint_{\mathcal{S}} \vec{D}\cdot d\vec{a} = Q_{f\,enc} \qquad (1)$$

closed surface

$$\vec{\nabla}\cdot\vec{B} = 0 \rightarrow \oint_{\mathcal{S}} \vec{B}\cdot d\vec{a} = 0 \qquad (2)$$

Stoke's theorem

$$\vec{\nabla}\times\vec{E} = -\frac{\partial \vec{B}}{\partial t} \rightarrow \oint_{P} \vec{E}\cdot d\vec{l} = -\frac{d}{dt}\int_{S} \vec{B}\cdot d\vec{a} \qquad (3)$$

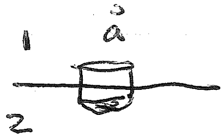
where any surface $\mathcal{S}$ bounded by closed loop P

$$\nabla\times\vec{H} = \vec{J}_f + \frac{\partial \vec{D}}{\partial t} \rightarrow \oint \vec{H}\cdot d\vec{l} = I_f + \frac{d}{dt}\int_{S} \vec{D}\cdot d\vec{a} \qquad (4)$$

Use eq 1 to get BC ~~on~~ for D

Gaussian pill box

sides don't cont.



$$(\vec{D}_1 - \vec{D}_2)\cdot\vec{a} = \sigma_f\, a$$

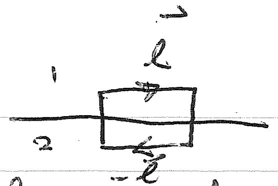
$$\rightarrow \quad D_1^{\perp} - D_2^{\perp} = \sigma_f$$

$^{\perp}$ means $\perp$ to $\vec{a}$

Similarly (2) $\rightarrow$ $B_1^{\perp} - B_2^{\perp} = 0$

side view

For (3)



sides → 0

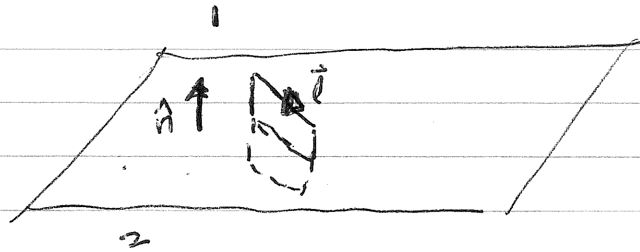
area → 0

$$\vec{E}_1 \cdot \vec{l} - \vec{E}_2 \cdot \vec{l} = -\frac{d}{dt}\int \vec{B} \cdot da \to 0$$

$$E_1^{\parallel} - E_2^{\parallel} = 0$$

For 4 $\quad \vec{H}_1 \cdot \vec{l} - \vec{H}_2 \cdot \vec{l} = I_{free}$

Need more than side view



I_{free} goes along surface $\quad$ no volume current

I_{free} is total current thru loop

need current thru loop $\quad$ magnitude is $K_f\, l$

direction is $\hat{n} \times \vec{l}$ $\qquad$ $I_{free} = \vec{K}_f \cdot (\hat{n} \times \vec{l}) = (\vec{K}_f \times \hat{n}) \cdot \vec{l}$

so $\quad \boxed{\vec{H}_1^{\parallel} - H_2^{\parallel} = \vec{K}_f \times \hat{n}}$