NAME: SOLUTIONS **Student ID:** **Score:**

Physics 322 **Winter 2018**

EXAM # 3
1030 AM - 1220 PM, Wednesday March 14

Write your name and ID number at the top of this page and on pages 2-5.

Clearly show all your reasoning.

You are not allowed to use calculators, computers or other programmable devices during the exam.

You are not allowed to use your phone during the exam.

This is a closed-book exam. Textbooks, class notes and other class material are not allowed.

Relevant formulae and equations are provided in a separate booklet and at the back of this booklet.

Clearly note all constants and assumptions you use.

Show all your work and your final answers in the spaces provided. If you need to use the back of a page to complete your answer, clearly indicate this. Scratch work will not be graded.

Extra paper is available at the front of the classroom.

If you have a question during the exam, raise your hand.

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1. (25 pts total) *Electromagnetic Wave in Linear Media*
Monochromatic laser radiation propagates in the $z-$direction through a non-magnetic linear medium (index of refraction n_1) with electric field at $(\mathbf{r}, t)$ described by $\mathbf{E} = E_1\cos(k_1 z - \omega_1 t)\hat{\mathbf{x}}$. $(\mathbf{r}, t)$ refers to position $\mathbf{r} = (x, y, z)$ and time t. Do not introduce any new variables for your answers below (fundamental constants ok).
(a) (5 pts) Write down an expression for the magnetic field of the wave at $(\mathbf{r}, t)$.

Linear medium, so the magnetic field $\mathbf{B}$ has amplitude $B_1 = E_1/v_1$ where the wave speed $v_1 = \omega_1/k_1 = c/n_1$. The $\mathbf{B}$ field direction is given by $\mathbf{E} \times \mathbf{B}$ pointing in the propagation direction $\hat{\mathbf{z}}$. Thus $\mathbf{B}$ points in the $\hat{\mathbf{y}}$ direction. The frequency and wavevector remain the same and $\mathbf{B}$ is in phase $\mathbf{E}$. Thus, we can write

$$\mathbf{B} = E_1\frac{k_1}{\omega_1}\cos(k_1 z - \omega_1 t)\hat{\mathbf{y}} = E_1\frac{n_1}{c}\cos(k_1 z - \omega_1 t)\hat{\mathbf{y}}$$

(b) (5 pts) What is the electromagnetic energy density u of the wave at $(\mathbf{r}, t)$?
Medium is non-magnetic, so $\mu = \mu_0$. Then, $n_1 = \sqrt{\mu\epsilon/\mu_0\epsilon_0} = \sqrt{\epsilon/\epsilon_0}$ and $\epsilon = \epsilon_0 n_1^2$. For the linear medium, we have (where E and B are the magnitudes):

$$u = \frac{\epsilon}{2}E^2 + \frac{1}{2\mu}B^2 = \epsilon E^2 = \epsilon_0 n_1^2 E_1^2\cos^2(k_1 z - \omega_1 t)$$

Thus $u = \epsilon_0 n_1^2 E_1^2\cos^2(k_1 z - \omega_1 t)$. Note that $n_1 = ck_1/\omega_1$ and $c = 1/\sqrt{\epsilon_0\mu_0}$ can be used to rewrite the $\epsilon_0 n_1^2$ in other acceptable ways.

(c) (5 pts) Write down an expression for the Poynting vector $\mathbf{S}$ at $(\mathbf{r}, t)$ and evaluate $\nabla \cdot \mathbf{S}$.

Poynting vector in the non-magnetic medium is $\mathbf{S} = \frac{1}{\mu_0}(\mathbf{E} \times \mathbf{B}) = \frac{E_1^2 k_1}{\mu_0\omega_1}\cos^2(k_1 z - \omega_1 t)\hat{\mathbf{z}}$ and $\nabla \cdot \mathbf{S} = \frac{\partial S_z}{\partial z} = -\frac{2E_1^2 k_1^2}{\mu_0\omega_1}\cos(k_1 z - \omega_1 t)\sin(k_1 z - \omega_1 t)$

(d) (5 pts) Evaluate $\frac{\partial u}{\partial t} + \nabla \cdot \mathbf{S}$ at $(\mathbf{r}, t)$. Explain in one sentence the result that you obtain.

Using result from (b), $\frac{\partial u}{\partial t} = 2\epsilon_0 n_1^2 E_1^2\omega_1\cos(k_1 z - \omega_1 t)\sin(k_1 z - \omega_1 t)$. Using (c) and noting that $\epsilon_0 n_1^2 = \frac{\epsilon_0 c^2 k_1^2}{\omega_1^2} = \frac{k_1^2}{\mu_0\omega_1^2}$ we have $\frac{\partial u}{\partial t} + \nabla \cdot \mathbf{S} = (2\epsilon_0 n_1^2\omega_1 - \frac{2k_1^2}{\mu_0\omega_1})E_1^2\cos(k_1 z - \omega_1 t)\sin(k_1 z - \omega_1 t) = 0$. Thus $\frac{\partial u}{\partial t} + \nabla \cdot \mathbf{S} = 0$. This is because of conservation of energy (Poynting's Theorem) in a source-free linear medium.

(e) (5 pts) Now suppose that this wave enters another non-magnetic linear medium (index of refraction n_2) at normal incidence. The resulting transmitted wave is described by $E_2\cos(k_2 z - \omega_2 t)\hat{\mathbf{x}}$. What are the values of k_2/k_1 and ω_2/ω_1 in terms of n_1 and n_2?

The frequency remains the same in order to match the boundary conditions on the phase of the wave and its temporal evolution. Thus $\omega_2/\omega_1 = 1$. Since both media are linear, we have $n_1 = ck_1/\omega_1$ and $n_2 = ck_2/\omega_2$, from discussions in (a) and (b). Therefore $k_2/k_1 = (\omega_2 n_2)/(\omega_1 n_1) = n_2/n_1$. Thus $k_2/k_1 = n_2/n_1$.

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2. (25 pts total) *Waves in a Conductor*
Copper is non-magnetic and a good conductor with $\frac{\sigma}{\epsilon\omega} \simeq 10^{12}$ and skin-depth $65\,\mu$m for $\omega = 2\pi \times 1\,$MHz radiation. Your final answers for (a-d) should be numerical values with units.

(a) (5 pts) What is the skin-depth in copper for $\omega = 2\pi \times 100\,$MHz?

Skin depth for good conductor $d = \frac{1}{Im(\tilde{k})} \simeq \sqrt{\frac{2}{\omega\sigma\mu}} \propto \frac{1}{\sqrt{\omega}}$

$d = 65\,\mu m$ at $\omega = 2\pi \times 1\,MHz$ (Cu)

$\Rightarrow d = 65 \times \sqrt{\frac{2\pi \times 1 MHz}{2\pi \times 100 MHz}}\,\mu m = 6.5\,\mu m$ at $2\pi \times 100 MHz$ in Cu

$\Rightarrow$ Skin depth is $6.5\,\mu m$

(b) (5 pts) What is the wavelength in copper for $\omega = 2\pi \times 100\,$MHz?

Wavelength $\lambda = \frac{2\pi}{Re(\tilde{k})} = 2\pi \times \sqrt{\frac{2}{\omega\sigma\mu}}$ for good conductor.

Using result of (a), $\lambda = 2\pi \times 6.5\,\mu m = 13\pi\,\mu m \simeq 40\,\mu m$.

$\Rightarrow \lambda$ in Cu at $2\pi \times 100\,MHz \simeq 40\,\mu m$

(c) (6 pts) What is the phase difference between the electric and magnetic fields of this wave in copper?

For wave inside conductor: $\tilde{\vec{E}} = \tilde{E}_0 e^{i(\tilde{k}z-\omega t)}\hat{x} = \tilde{E}_0 e^{-Kz} e^{i(kz-\omega t)}\hat{x}$

as an example of wave propagating in the z-direction. $\tilde{\vec{B}} = \tilde{B}_0 e^{i(\tilde{k}z-\omega t)}\hat{y} = \tilde{B}_0 e^{-Kz} e^{i(kz-\omega t)}\hat{y}$

$[\tilde{k} = k + iK]$

From Faraday's law: $\vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$, we get: $\tilde{k}\tilde{E}_0 = \omega \tilde{B}_0$

$\tilde{k}$ can be written as $Ke^{i\phi}$ and $\vec{B}$ lags $\vec{E}$ by ϕ.

For a good conductor $\phi = \arctan\left(\frac{K}{k}\right) \simeq \arctan(1) = 45^\circ$. $\Rightarrow \vec{B}$ lags $\vec{E}$ by 45°

(d) (9 pts) What is the ratio of the (time-averaged) energy stored in the magnetic field to that in the electric field for this wave inside copper?

(Time average) Energy in electric field $= \frac{1}{2}\epsilon\langle \vec{E}^2 \rangle = \frac{1}{4}\epsilon|\tilde{E}_0|^2$ time average over $\cos^2$ gives $\frac{1}{2}$

(Time average) Energy in magnetic field $= \frac{1}{2\mu_0}\langle \vec{B}^2 \rangle = \frac{1}{4\mu_0}|\tilde{B}_0|^2$ Cu is non-magnetic

Ratio of energies $= \frac{|\tilde{B}_0|^2}{|\tilde{E}_0|^2}\frac{1}{\epsilon\mu_0} = \left|\frac{\tilde{k}}{\omega}\right|^2 \frac{1}{\epsilon\mu_0}$ (from result in (c))

$= \frac{k^2+K^2}{\omega^2}\frac{1}{\epsilon\mu_0} = \frac{\omega\sigma\mu_0}{\omega^2}\frac{1}{\epsilon\mu_0} = \frac{\sigma}{\epsilon\omega}$

[for good conductor $k = K = \sqrt{\frac{\omega\sigma\mu}{2}}$]

Thus, ratio of stored energy $\simeq 10^{12}$ for $\omega = 2\pi \times 1\,MHz$

$\simeq 10^{10}$ for $\omega = 2\pi \times 100\,MHz$

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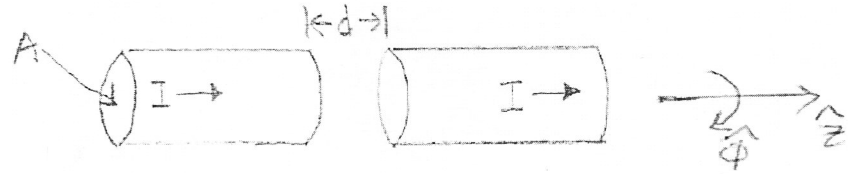
3. (25 pts total) *Current-carrying wire with gap*

A large diameter wire of cross-section area A carries current uniformly over its cross section. There is a narrow gap of width d, forming a parallel-plate capacitor, as shown. The current is zero for times $t < 0$ and the current is I at times $t > 0$. The charge on the capacitor is zero at $t = 0$. Assume $t > 0$ for the following and neglect fringe fields.



(a) (4 pts) Find the electric field in the gap (magnitude and direction).

charge on capacitor plates at time t (>0) is $\pm It$.

Thus, electric field in gap $\vec{E} = \frac{(It/A)}{\epsilon_0}\hat{z} = \frac{It}{A\epsilon_0}\hat{z}$

where $\hat{z}$ on cylinder axis pointing to the right (as in figure).

(b) (5 pts) Find the magnetic field in the gap (magnitude and direction).

For Amperian loop in gap with radius s (less than wire radius), we have: $B(s)2\pi s = \mu_0 I_{enc} + \mu_0\epsilon_0 \int \frac{\partial \vec{E}}{\partial t}\cdot d\vec{a}$ (Ampere-Maxwell)

$I_{enc} = 0$ in gap and $\frac{\partial \vec{E}}{\partial t} = \frac{I}{A\epsilon_0}\hat{z} \Rightarrow \vec{B} = \mu_0\epsilon_0 \frac{I}{A\epsilon_0}\frac{\pi s^2}{2\pi s}\hat{\phi} = \frac{\mu_0 I}{2A} s\,\hat{\phi}$

[$\hat{\phi}$ as in figure].

(c) (5 pts) Find the electromagnetic energy density u in the gap.

$$u = \frac{1}{2}\epsilon_0 E^2 + \frac{1}{2\mu_0}B^2 = \frac{1}{2}\epsilon_0 \frac{I^2t^2}{A^2\epsilon_0^2} + \frac{1}{2\mu_0}\frac{\mu_0^2 I^2 s^2}{4A^2}$$

$$\Rightarrow u = \frac{1}{2}\frac{I^2t^2}{\epsilon_0 A^2} + \frac{1}{8}\frac{\mu_0 I^2 s^2}{A^2}$$

(d) (5 pts) Find the momentum density $\mathbf{g}$ in the gap (magnitude and direction).

$$\vec{g} = \epsilon_0(\vec{E}\times\vec{B}) = \epsilon_0 \frac{It}{A\epsilon_0}\hat{z} \times \frac{\mu_0 I s}{2A}\hat{\phi} = \frac{\mu_0 I^2 t s}{2A^2}(\hat{z}\times\hat{\phi})$$

$$\Rightarrow \vec{g} = \frac{\mu_0 I^2 ts}{2A^2}(-\hat{s}) = -\frac{\mu_0 I^2 ts}{2A^2}\hat{s}$$

points towards the cylinder axis (inwards)

(e) (6 pts) What is the value of $\nabla \cdot \mathcal{T}$ in the gap, where $\mathcal{T}$ is the Maxwell stress tensor. Explain your answer.

There are no charges in the gap, to the conservation of momentum takes the form $\frac{\partial \vec{g}}{\partial t} = \vec{\nabla}\cdot\overleftrightarrow{T}$

Thus, $\vec{\nabla}\cdot\overleftrightarrow{T} = -\frac{\mu_0 I^2 s}{2A^2}\hat{s}$

This is a statement of conservation of momentum and equivalent to a "continuity" equation for electromagnetic momentum.

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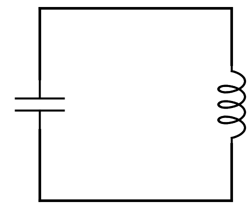
IV. [25 points total] This question consists of two independent parts: A and B.

A. Consider an LC circuit. Recall that LC circuits oscillate with period τ. At t =0, the top plate of the capacitor has maximum positive charge, and the current throughout the circuit is zero.

At $t = 0$
$Q_{top} = +$
$I = 0$



i. [6 pts] At t = $\tau/8$, where is the divergence of the Poynting vector, $\vec{\nabla} \cdot \vec{S}$, positive? Explain your reasoning.

At t = 0, there was no current but the top plate of the capacitor is positively charged. At t = $\tau/8$, the charge on the top plate is still positive but decreasing, and the current is clockwise and increasing. One method to find where $\vec{\nabla} \cdot \vec{S}$ is positive is to think about where electromagnetic energy is being transferred from. At this time, the energy is flowing from the capacitor to the inductor since the electric field between the capacitor is decreasing, so the region of positive $\vec{\nabla} \cdot \vec{S}$ is between the capacitor plates, or at the capacitor. This is the verbal way of using $\vec{\nabla} \cdot \vec{S} = -\frac{du}{dt}$. Also, the electric field in the capacitor is pointing down, and the magnetic field around the capacitor is counter-clockwise as viewed from above, so the cross product from $\vec{S} = \frac{1}{\mu_o}(\vec{E} \times \vec{B})$ is out from the capacitor, which is positive divergence.

ii. [6 pts] In terms of τ, what is the period of oscillation for the Poynting vector? If the Poynting vector is constant, state so explicitly. Explain your reasoning.

The period of the Poynting vector relative to the period of the LC oscillation is best determined qualitatively by thinking of it as energy flow. In the time that energy flows from the capacitor to the inductor and then back to the capacitor, the capacitor's charge is flipped such that there is net negative charge on the top. Thus, the period of the Poynting vector is half the period of the LC oscillation, or $\tau/2$. You can show this mathematically from $\vec{S} = \frac{1}{\mu_o}(\vec{E} \times \vec{B})$ by saying that both fields oscillate with cos(ωt) (or sin), so the cross product will oscillate with cos^2(ωt). This can be expanded to ½ + ½ cos(2ωt), so twice the frequency means half the period.

B. Consider the following expression: $\tilde{E} = E_o e^{i(2x-3y+6t)}(\hat{z})$

i. [3 pts] Determine the (*x, y*)-coordinates of a point at *t* = 0 for which the argument of the complex exponential is 0, other than the origin. Plot that point on the graph at right and label it as *P*. *(Sorry, issue with graphic.)*

Set the argument to be zero at t = 0, so 2x-3y=0. Thus any point along the line y = +(2/3)x works. The easiest one to see is (3, 2).

ii. [5 pts] Consider the line passing through point *P* and the origin. Is that line **parallel** to the direction of *propagation*, *polarization*, *both*, or *neither?* Explain your answer

Anywhere that has the same argument is one of the planes of the plane wave, so it must be perpendicular to the direction of propagation. The polarization of the electric field is in the z-direction, which is perpendicular to anything drawn. Thus it is parallel to neither.*

**If you explicitly said that it is parallel to the polarization of the magnetic field, that is correct.*

iii. [5 pts] Does the propagation direction of this wave have a component in the *positive y-direction, negative y-direction,* or *neither* (*i.e, upwards, downwards,* or *horizontal)?* Explain your reasoning.

By looking at the argument, as time increases, the argument also increases. Thus in order to follow the wave, the y-value must also increase. Thus, the propagation direction has an upward component. If you look at the way the argument is conventionally written as $\pm i(\vec{k} \cdot \vec{r} - \omega t)$, you can see that $\vec{k} = -2\hat{x} + 3\hat{y}$, which has an upward or positive y-component. You can also use a mix of these reasonings ($\vec{k} = +2\hat{x} - 3\hat{y}$ but propagates backwards because of the $+\omega t$), or plot a plane at t = 1 and see the upward shift.