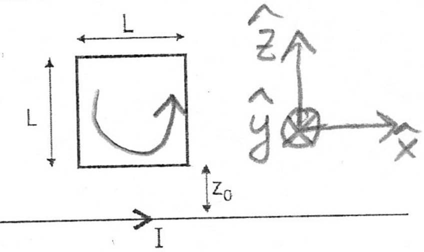
NAME: SOLUTIONS **Student ID:** Score:

1. (25 pts total) *Induced EMF*

An infinitely long thin wire is placed along the x-axis and carries current I in the $+\hat{x}$ direction. A square loop of side length L, mass m and resistance R, is placed in the xz-plane centered at $(0, 0, z_0 + L/2)$, and with its sides parallel to the x and z axes. Ignore self inductance.

(a) (6 pts) What is the magnetic flux through the loop?



Using Ampere's Law, field due to wire carrying current I is $\vec{B} = \frac{\mu_0 I}{2\pi z}(-\hat{y})$, points out of page inside loop.

Flux $\Phi_m = \int \vec{B}\cdot d\vec{a} = \frac{\mu_0 I}{2\pi}\int_{z_0}^{L+z_0} \frac{L\,dz}{z} = \frac{\mu_0 I L}{2\pi}\ln\left(\frac{L+z_0}{z_0}\right)$

An external agent gives the loop a sudden push in the $+\hat{z}$ direction and the loop acquires an initial velocity $v(t = 0) = v_0$ in the $+\hat{z}$ direction.

(b) (6 pts) What is the initial EMF induced in the loop?

$$\text{EMF} = -\frac{d\Phi_m}{dt} = -\frac{\mu_0 I L}{2\pi}\frac{d}{dt}\left[\ln(L+z) - \ln z\right]$$ at time t when loop is centered at $(0,0,z+\frac{L}{2})$

$$= -\frac{\mu_0 I L}{2\pi}\left[\frac{dz/dt}{L+z} - \frac{dz/dt}{z}\right] = \frac{\mu_0 I L}{2\pi}v\left[\frac{L}{z(z+L)}\right] = \frac{\mu_0 I L^2 v}{2\pi z(z+L)}$$

where v is upward velocity at time t.

$$\Rightarrow \text{Initial EMF } (t=0) = \mathcal{E}_0(v=v_0, z=z_0) = \frac{\mu_0 I L^2 v_0}{2\pi z_0(z_0+L)}$$

(c) (5 pts) What is the magnitude of the initial current that flows through the loop? Indicate its direction clearly on the figure.

Initial current through loop $= \frac{\text{initial EMF}}{R} = \frac{\mu_0 I L^2 v_0}{2\pi z_0(z_0+L)}\cdot\frac{1}{R}$

The motion reduces the magnetic flux coming out of page through the loop.

Thus, by Lenz's Law, the induced current flows counterclockwise, as shown in figure.

(d) (8 pts) Derive a differential equation for the time evolution of the loop velocity $v(t)$. The only variables it should contain are v and t. You do not need to solve this equation.

The loop feels a magnetic force on each of the horizontal sections of the form $I_{loop}\int d\vec{\ell}\times\vec{B}$. The direction is upwards for the upper section and downwards for the lower section. The net force is downwards because $\vec{B}$ is larger at smaller z. Thus, we have:

$$F_{loop} = m\frac{dv}{dt} = -I_{loop}L\left[\frac{\mu_0 I}{2\pi z} - \frac{\mu_0 I}{2\pi(z+L)}\right] = -\frac{I_{loop} I L \mu_0}{2\pi}\frac{L}{z(z+L)}$$

(in $+\hat{z}$ direction)

$$\Rightarrow m\frac{dv}{dt} = -\frac{\mu_0 I L^2}{2\pi z(z+L)}\,\frac{\mu_0 I L^2 v}{2\pi z(z+L)}\cdot\frac{1}{R} = -\left(\frac{\mu_0 I}{2\pi}\right)^2\frac{L^4 v}{R}\frac{1}{z^2(z+L)^2}$$

$$\Rightarrow \frac{dv}{dt} = -\left(\frac{\mu_0 I}{2\pi}\right)^2\frac{L^4 v}{mR}\frac{1}{z^2(z+L)^2}$$

[at time t, loop centered at $(0,0,z+L/2)$ as in (b)]

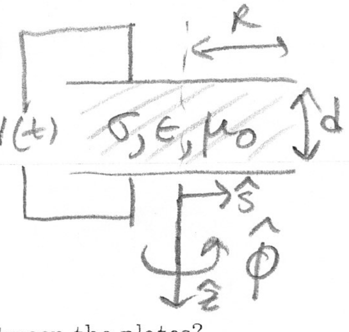
NAME: SOLUTIONS Student ID: Score:

2. *Conduction and Displacement current*

The space between the electrodes of a parallel-plate capacitor is filled with a uniform medium with conductivity σ, permittivity ϵ, and permeability μ_0. The horizontally-oriented capacitor plates are circular with radius R, and are spaced apart by distance d. Here $d \ll R$ and you can neglect fringe fields. An experimenter connects the lower plate to ground and starting at time $t = 0$, applies a (slowly-)varying voltage $V(t) = V_0 + \alpha t$, to the upper plate. Each of V_0 and α can be positive or negative.

(a) (6 pts) What is the magnitude of the conduction current density $J(t)$ between the plates?

Electric field across plates $= \frac{V(t)}{d} = \frac{V_0 + \alpha t}{d} = E(t)$ and points in the vertical direction



$J(t) = \sigma E(t) = \frac{\sigma}{d}(V_0 + \alpha t)$

magnitude of conduction current density $J(t) = \frac{\sigma}{d}(V_0 + \alpha t)$

(b) (6 pts) What is the magnitude of the displacement current density $J_d(t)$ between the plates?

Displacement current density $J_d(t) = \epsilon \frac{\partial E}{\partial t} = \epsilon \frac{1}{d}(\alpha)$ and points in vertical direction

$\Rightarrow$ magnitude of $J_d(t) = \frac{\epsilon \alpha}{d}$

The experimenter finds that at time $t = t_0$, there is no electric field between the plates.

(c) (5 pts) Determine the relationship between V_0, α and t_0.

At $t = t_0$, $E(t_0) = \frac{V_0 + \alpha t_0}{d} = 0 \Rightarrow V_0 + \alpha t_0 = 0$

or, $t_0 = -\frac{V_0}{\alpha}$

(d) (8 pts) Find the magnetic field (magnitude and direction) between the plates at time $t = t_0$.

At $t = t_0$, $J(t_0) = 0$ and $J_d(t_0) = \frac{\epsilon \alpha}{d}$

from Maxwell's equation: $\vec{\nabla} \times \vec{B} = \mu_0 \vec{J} + \mu_0 \vec{J}_d$

At $t = t_0$ $\vec{\nabla} \times \vec{B} = \frac{\mu_0 \epsilon \alpha}{d}$

Apply Stokes law to get to integral form and evaluate for circular loop radius s within plates and coaxial with them.

By symmetry, $\vec{B} = B(s)\hat{\phi}$. $\Rightarrow \oint \vec{B} \cdot d\vec{l} = \int \frac{\mu_0 \epsilon \alpha}{d} da = \frac{\mu_0 \epsilon \alpha}{d} \pi s^2$

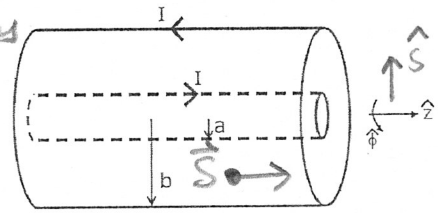
$\Rightarrow B(s) \cdot 2\pi s = \frac{\mu_0 \epsilon \alpha}{d} \pi s^2 \Rightarrow \vec{B} = \frac{\mu_0 \epsilon \alpha}{2d} s \hat{\phi}$, $\hat{\phi}$ is azimuthal direction (see figure).

NAME: SOLUTIONS Student ID: Score:

3. *Coaxial Cable*

As shown in the figure, a long coaxial cable consists of two very long, thin cylindrical conducting tubes (radii a, b with $a < b$), separated by non-conducting material (σ=0) with permittivity ϵ_0 and permeability μ_0. Current I flows along the inner conductor and returns along the outer one; in each case, the current distributes itself uniformly over the surface.

(a) (4 pts) Find the magnetic field (magnitude and direction) in the region between the tubes.



Use Ampere's law on circular loop radius s concentric with cylinders, $a<s<b$.

By symmetry $\vec{B}=B(s)\hat{\phi}$ and $\oint \vec{B}\cdot d\vec{l} = \mu_0 I_{enc}$

$\Rightarrow B(s)\,2\pi s = \mu_0 I \Rightarrow \underline{\vec{B} = \frac{\mu_0 I}{2\pi s}\hat{\phi}}$

(b) (5 pts) Find the magnetic energy stored in a length l of cable and use your answer to determine the inductance per unit length L/l.

Magnetic energy stored in length l $= U_m = \int \frac{B^2}{2\mu_0} d\tau = \frac{1}{2\mu_0}\frac{\mu_0^2 I^2}{(2\pi)^2}\left(\int_a^b \frac{2\pi s\, ds}{s^2}\right) l$.

$\Rightarrow U_m = \frac{\mu_0 I^2}{4\pi} l \int_a^b \frac{ds}{s} = \frac{\mu_0 I^2 l}{4\pi}\ln(b/a)$. Equating this to energy stored in an inductor $= \frac{1}{2}LI^2$, we get $\underline{\frac{L}{l} = \frac{\mu_0}{2\pi}\ln(b/a)}$

Now suppose that in addition, the conductors are held at a potential difference V, the inner conductor being at higher potential.

(c) (6 pts) Find the electric field (magnitude and direction) in the region between the tubes.

Suppose charge per unit length $+\lambda$ on inner cylinder. Apply Gauss' Law on cylindrical surface radius s, length l, concentric with tubes, $a<s<b$. By symmetry $\vec{E}=E(s)\hat{s}$ and

$E(s)\,2\pi s l = \lambda l/\epsilon_0 \Rightarrow E(s) = \frac{\lambda}{2\pi\epsilon_0}\frac{1}{s}$. Since $V(a)-V(b)=V=-\int_b^a \vec{E}\cdot d\vec{l}$

Then, $V = \frac{-\lambda}{2\pi\epsilon_0}\int_b^a \frac{ds}{s} = \frac{\lambda}{2\pi\epsilon_0}\ln\left(\frac{b}{a}\right) \Rightarrow \lambda = \frac{2\pi\epsilon_0 V}{\ln(b/a)} \Rightarrow \vec{E} = \frac{2\pi\epsilon_0 V}{\ln(b/a)}\cdot\frac{1}{2\pi\epsilon_0}\frac{1}{s}\hat{s} \Rightarrow \underline{\vec{E} = \frac{V}{\ln(b/a)}\frac{\hat{s}}{s}}$

(d) (5 pts) Determine the magnitude of the Poynting vector in the region between the tubes. Clearly indicate its direction in the figure.

$\vec{S} = \frac{1}{\mu_0}(\vec{E}\times\vec{B})$ $\quad$ $\vec{E}$ points along $\hat{s}$, $\vec{B}$ points along $\hat{\phi} \Rightarrow \underline{\vec{E}\times\vec{B} \text{ along } \hat{z}}$

$|\vec{S}| = \frac{1}{\mu_0}|\vec{E}||\vec{B}| = \frac{1}{\mu_0}\frac{V}{\ln(b/a)}\cdot\frac{1}{s}\cdot\frac{\mu_0 I}{2\pi s} = \underline{\frac{VI}{2\pi\ln(b/a)}\cdot\frac{1}{s^2}}$ $\quad$ Direction along $\hat{z}$ as shown in figure

($\vec{E}$ is perp^r to $\vec{B}$)

(e) (5 pts) By integrating the Poynting vector over a cross-section of the cable, determine the power transported through the cable. Express your answer in terms of the provided quantities.

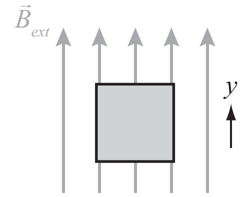
$\int \vec{S}\cdot d\vec{a} = \frac{VI}{2\pi\ln(b/a)}\cdot\int_a^b \frac{1}{s^2}\,2\pi s\, ds = \frac{VI}{\ln(b/a)}\int_a^b \frac{ds}{s} = \frac{VI}{\ln(b/a)}\ln(b/a)$

$\Rightarrow$ Power transported $= \int \vec{S}\cdot d\vec{a} = \underline{\underline{VI}}$ as expected.

Name ______________________________________ **Student ID** _______________ **Score**________
last *first*

IV. [25 points total] Tutorial question.

Consider a **para**magnetic cube placed in a uniform field pointing toward the top of the page, in $+\hat{y}$. The field may be expressed as $\vec{B}_{ext} = B_o\hat{y}$ and $\vec{H}_{ext} = H_o\hat{y}$. You may assume that the material is linear, but $|\chi_M|$ is not small, such that $|\chi_M| \sim 1$.



Paramagnetic cube in uniform external field

A. [2 pts.] In the box below, draw an arrow indicating the direction of the magnetization at the center of the cube. You do not need to explain.

Up.

B. [5 pts.] In the box below, draw an arrow indicating the direction of the net magnetic force exerted on the cube. If the net magnetic force is zero, state so explicitly. Explain your reasoning.

There is no net force.

You can think of this uniform field as being created by a large north pole below and a large south pole above. Any pole of the paramagnet would experience a pair of equal and opposite forces from the external poles in this approximation, so the net force is zero. This is also described by the relationship $\vec{F} = \vec{\nabla}(\vec{m} \cdot \vec{B})$ where the dot product is the same no matter where you put the cube, and thus has no derivative.

C. The following two questions may be answered independently of the other. You may use one part to help answer the other part, but please avoid circular logic. Keep in mind that $|\chi_M| \sim 1$.

i. [9 pts.] At the center of the cube, is the magnitude of the net magnetic field $|\vec{B}_{net}|$ *greater than, less than,* or *equal to* B_o from the external field? Explain your reasoning.

Given that the magnetization points up, we can deduce that the bound current goes around the cube (counterclockwise from above with $\vec{\nabla}\times\vec{M} = \vec{J}_{bound}$). The cube can influence $\vec{B}_{net}$ with the bound current since $\vec{\nabla}\times\vec{B} = \mu_o\vec{J}_{net}$ but $\vec{\nabla} \cdot \vec{B} = 0$. Using the right hand rule, $\vec{B}_{ind}$ is up within the cube, so $|\vec{B}_{net}|$ is greater than the original B_o.

ii. [9 pts.] At the center of the cube, is the magnitude of the net auxiliary field $|\vec{H}_{net}|$ *greater than, less than,* or *equal to* H_o from the external field? Explain your reasoning.

Again, given that the magnetization points up, we can see that there is negative $\vec{\nabla} \cdot \vec{M}$ on the top surface of the cube, and positive $\vec{\nabla} \cdot \vec{M}$ at the bottom of the cube (North and South poles respectively). These can influence $\vec{H}_{net}$ because $\vec{\nabla} \cdot \vec{H} = -\vec{\nabla} \cdot \vec{M}$, although $\vec{J}_{bound}$ does not as $\vec{\nabla}\times\vec{H} = \vec{J}_{free}$. Consequently, $\vec{H}_{ind}$ points downward, away from positive divergence and toward negative divergence, so $|\vec{H}_{net}|$ is less than the original H_o.

n.b., Variants of the equation $\vec{B} = \mu\vec{H}$ or $\vec{H} = \vec{B}/\mu_o - \vec{M}$ are not helpful here, because with $\mu_o < \mu \sim 2\mu_o$ all three variables change significantly from the external case to the net case. For instance, if you know that both $B_o < B_{net}$ and $\mu_o < \mu$, you cannot compare H_{net} and H_o without knowing values. You need the approximation that $|\chi_M| << 1$ to do anything productive with this relationship. However, this can earn full consistency points if one mistakenly showed that either H or B stayed the same.