1. (25 pts total) **Concentric circular coils**

You are provided the information that the on-axis magnetic field of a circular loop of radius $R$, carrying current $I$, centered at the origin, and placed parallel to the $xy$ plane points up or down the $z$-axis (according to the right hand rule) and has magnitude $B(z) = \frac{\mu_0 I}{2R \sqrt{1 + (z/R)^2}}$. 

Now consider two such loops, both centered at the origin and parallel to the $xy$ plane, but with different radii $R$ and $2R$, carrying currents $I_1$ and $I_2$ respectively, and in opposite directions, with $I_1$ circulating in the $\hat{z}$ direction.

(a) (5 pts) What is the net magnetic field at the origin?

By superposition principle, at origin, $B_x = B_y = 0$ and $B_z = \frac{\mu_0 I_1}{2R} - \frac{\mu_0 I_2}{4R}$.

Thus, $B = \frac{\mu_0}{4R} (I_1 - I_2) \hat{z}$ at origin.

(b) (8 pts) An experimenter wants to create a local magnetic field zero (ie $B = 0$) above the $xy$ plane at the location $z = R$ on the $z$-axis. Find the ratio of $I_1/I_2$ that she must satisfy to make this happen. Ignore the trivial $I_1 = I_2 = 0$ case. If it is not possible to do this, explain why.

At location $(x, y, z) = (0, 0, R)$, $B_x = B_y = 0$.

$$B_z = \frac{\mu_0 I_1}{2R} (1 + \frac{1}{16}) - \frac{\mu_0 I_2}{4R} (1 + \frac{1}{2}) \left(1 + \frac{1}{4}\right) = \frac{\mu_0 I_1}{4R} \left[\frac{1}{\sqrt{2}} - \frac{I_2}{(\frac{5}{4})^{3/2}}\right]$$

For $B = 0$ at $(0, 0, R)$, she must satisfy:

$$\frac{I_1}{\sqrt{2}} = \frac{I_2}{(\frac{5}{4})^{3/2}} \quad \Rightarrow \quad \frac{I_1}{I_2} = \frac{\sqrt{2} 4^{3/2}}{5^{3/2}} = \frac{8\sqrt{2}}{5\sqrt{5}}$$

(c) (6 pts) For arbitrary $I_1$ and $I_2$, find an expression to lowest order in $1/z$ for the on-axis magnetic field for large $z$ (ie, $z \gg 2R$). Simplify as much as you can.

For $z \gg 2R$, $(\frac{z}{R})^2$ and $(\frac{2R}{z})^2 \gg 1$, so at location $(0, 0, z)$ again $B_x = B_y = 0$ and $B_z \approx \frac{\mu_0 I_1}{2R} \left(\frac{z}{2R}\right)^2 - \frac{\mu_0 I_2}{4R} \left(\frac{z}{2R}\right)^3$

$$\Rightarrow \quad B \approx \frac{\mu_0}{2R} (I_1 R^2 - 4 I_2 R^2) \hat{z} \quad \text{for large } z, \text{ to lowest order in } \frac{1}{z}$$

(d) (6 pts) Now suppose the experimenter wants to make this lowest order field vanish at large $z$. What is the ratio of currents $I_1/I_2$ that she must apply? Connect your result to the magnetic dipole moment of each current loop.

To make $\frac{1}{2\hat{z}}$ field vanish, must have $I_1 R^2 = 4 I_2 R^2$

$$\Rightarrow \quad \frac{I_1}{I_2} = 4$$

Magnetic dipole moment of loops one $m_1 = I_1 \pi R^2 \hat{z}$ and $m_2 = -I_2 \pi (2R)^2 \hat{z}$

To make $\frac{1}{2\hat{z}}$ component (ie dipole component) vanish, total dipole moment must be zero. That is $m_1 + m_2 = 0$

$$\Rightarrow \quad I_1 \pi R^2 - I_2 \pi (2R)^2 = 0 \quad \Rightarrow \quad \frac{I_1}{I_2} = 4, \text{ as found earlier.}$$
2. \textit{Thick Solenoid}

A thick infinitely-long solenoid (see figure) carries a uniform volume current density \( \mathbf{J} = J_0 \delta \) between its inner and outer radii \( a \) and \( b \). \( J_0 \) is a constant, there are no other currents, and assume that the material is non-magnetic.

(a) (5 pts) Find the magnetic field \( \mathbf{B} \) (magnitude and direction) outside the solenoid (ie, at distances \( s > b \), where \( s \) is the radial distance from the solenoid axis.)

By symmetry, \( \mathbf{B} = B(s) \hat{z} \).

Applying Ampère's law \( \oint \mathbf{B} \cdot d\mathbf{l} = \mu_0 I \) to rectangular loop 1 (see figure) with height \( L \) and vertical sides located at \( s \) and \( s' > b \), we get:

\[ B(s)L - B(s')L = 0 \Rightarrow B(s) = B(s') \text{.} \]

Since \( B \) must approach zero far away solenoid, \( B = 0 \) outside \( s > b \).

(b) (5 pts) Find \( \mathbf{B} \) within the solenoid material \( (a < s < b) \).

Again by symmetry, \( \mathbf{B} = B(s) \hat{z} \) and applying to rectangular loop 2 (see figure) with \( a < s < b \) and \( s' > b \), we get:

\[ B(s)L - B(s')L = \mu_0 J_0 (b-s)L \]

From part (a), \( B(s') = 0 \Rightarrow B(s) = \mu_0 J_0 (b-s) \Rightarrow B = \mu_0 J_0 (b-s) \hat{z} \text{ for } a < s < b \).

(c) (5 pts) Find \( \mathbf{B} \) inside the solenoid \( (s < a) \).

Again, \( \mathbf{B} = B(s) \hat{z} \) and applying Ampère's law to loop 3 (see figure) with \( s < a \) and \( s' > b \), we get:

\[ B(s)L - B(s')L = \mu_0 J_0 (b-a)L \cdot \]

From part (a), \( B(s') = 0 \Rightarrow B(s) = \mu_0 J_0 (b-a) \Rightarrow B = \mu_0 J_0 (b-a) \hat{z} \text{ for } s < a \).

(d) (10 pts) Find the vector potential \( \mathbf{A} \) (magnitude and direction) everywhere. Take \( \mathbf{A} = 0 \) at \( s = 0 \) ie, on the solenoid axis. Note that the potential must be continuous.

By symmetry, \( \mathbf{A} = A(s) \hat{z} \).

We will apply \( \oint \mathbf{A} \cdot d\mathbf{l} = \Phi_m = \int \mathbf{B} \cdot d\mathbf{a} \) to a circular loop radius \( s \), concentric with the solenoid and enclosing

When \( s < a \):
\[ A(s)2\pi s = \mu_0 J_0 (b-a) \pi s^2 \Rightarrow A(s) = \frac{\mu_0 J_0 (b-a) s}{2} \] \( \text{[s < a]} \).

When \( a < s < b \):
\[ A(s)2\pi s = 2\mu_0 J_0 \pi \frac{b-a}{2} + \int_a^s \mu_0 J_0 \pi (b-s) d(s) \]
\[ = \mu_0 J_0 \pi \left[ \frac{b-s^2}{2} \pi s^2 - \frac{1}{3} (b-a)^3 \right] \]
\[ = \mu_0 J_0 \pi \left[ \frac{bs^2}{2} - \frac{b^3}{2} - \frac{1}{3} (b-a)^3 \right] \]
\[ = A(s) = \frac{\mu_0 J_0}{2} \left[ b s^2 - \frac{1}{2} s^3 + \frac{1}{3} (b-a)^3 \right] \text{ for } a < s < b \).

When \( s > b \):
\[ A(s)2\pi s = \mu_0 J_0 \pi \left[ b, b^2 - \frac{b^3}{2} - \frac{1}{3} (b-a)^3 \right] = \frac{\mu_0 J_0 \pi (b^3-a^3)}{3} \]
\[ \Rightarrow A(s) = \frac{\mu_0 J_0 (b^3-a^3)}{6} \text{ for } s > b \).

In each region \( \mathbf{A} = A(s) \hat{z} \).
3. Coaxial Cable
As shown in the figure, a coaxial cable consists of two very long, thin cylindrical tubes (radii \(a\) and \(b\)), separated by linear insulating material of magnetic susceptibility \(\chi_m\). A free current \(I\) flows along the inner conductor and returns along the outer one; in each case, the current distributes itself uniformly over the surface.

(a) (8 pts) Use Ampere's law for \(\mathbf{H}\) to find the magnetic field (magnitude and direction) in the region between the tubes.

By symmetry of the problem, \(\mathbf{H} = H(s) \mathbf{\hat{\phi}}\).

Applying Ampere's law \(\oint \mathbf{H} \cdot d\mathbf{l} = I_{\text{enc}}\) to a circular loop radius \(s\) \((a < s < b)\) and concentric with the tubes:

\[
H(s) \cdot 2\pi s = I_{\text{enc}} \Rightarrow H(s) = \frac{I_{\text{enc}}}{2\pi s} = \frac{I}{2\pi s} \mathbf{\hat{\phi}}
\]

\[
\Rightarrow \mathbf{M} = \chi_m H = \chi_m \frac{I}{2\pi s} \mathbf{\hat{\phi}}
\]

\[
\Rightarrow \mathbf{B} = \mu_0 (M + \mathbf{H}) = \mu_0 (1 + \chi_m) \frac{I}{2\pi s} \mathbf{\hat{\phi}}
\]

in the region between the tubes \((a < s < b)\).

(b) (9 pts) Find the bound currents in the insulator.

Bound volume current \(\mathbf{J}_b = \nabla \times \mathbf{M} = \frac{1}{s} \frac{d}{ds} (s \chi_m \frac{I}{2\pi s})\)

\[
\Rightarrow \mathbf{J}_b = \frac{1}{s} \frac{d}{ds} \left[ s \chi_m \frac{I}{2\pi s} \right] = 0
\]

in the insulator, which makes sense because the insulator is linear and there are no free currents within it.

Bound surface current \(\mathbf{K}_b = \mathbf{M} \times \mathbf{\hat{n}}\) at the two surfaces.

\[
\begin{align*}
\text{At } s = a: \quad & \mathbf{K}_b = \chi_m \frac{I}{2\pi a} \mathbf{\hat{\phi}} \times \mathbf{\hat{n}} = \chi_m \frac{I}{2\pi a} \mathbf{\hat{\phi}} \\
\text{At } s = b: \quad & \mathbf{K}_b = \chi_m \frac{I}{2\pi b} \mathbf{\hat{\phi}} \times \mathbf{\hat{n}} = \chi_m \frac{I}{2\pi b} \mathbf{\hat{\phi}}
\end{align*}
\]

(c) (8 pts) Using your answer to (b), the free currents, and Ampere's law for \(\mathbf{B}\), find the magnetic field (magnitude and direction) in the region between the tubes.

By symmetry, \(\mathbf{B} = B(s) \mathbf{\hat{\phi}}\). Applying Ampere's law \(\oint \mathbf{B} \cdot d\mathbf{l} = \mu_0 I\) to circular loop radius \(s\) \((a < s < b)\) and concentric with the tubes:

\[
B(s) \cdot 2\pi s = \mu_0 I_{\text{total}} = \mu_0 \left[ I + \chi_m \frac{I}{2\pi a} \right]
\]

\[
\Rightarrow B(s) = \mu_0 (1 + \chi_m) \frac{I}{2\pi s}
\]

\[
\Rightarrow \mathbf{B} = \mu_0 (1 + \chi_m) \frac{I}{2\pi s} \mathbf{\hat{\phi}}
\]

in agreement with the result in (a).

A non-magnetic slab with uniform current density $\vec{J}_o$ directed out of the page ($\sim \hat{y}$) fills the space from $z = -d$ to $z = d$. The slab is infinite in both $x$ and $y$. The magnetic field at the origin (Point $O$) is zero. There are no other sources in this system.

A. [7 pts.] What component(s) of the magnetic field due to this slab at point $P$ must be zero? For each component you identified, explain your reasoning.

- **y-component**: Biot-Savart Law shows that every $d\vec{B}$ from every small current element must be perpendicular to that current element. Since all of the current elements only point in $\hat{y}$, the magnetic field cannot have a component in that direction.

- **z-component**: A $180^\circ$ rotational symmetry about any $y$-axis $z = 0$ shows that a magnetic field with a z-component above the slab must have the opposite z-component below the slab. This is inconsistent with $\nabla \cdot \vec{B} = 0$, so the magnetic field cannot have a z-component. One could also argue that the superposition of the z-component, as wires from the right side of the slab will have a downward component while wires from the left side will have an equally strong upward component.

B. [10 pts.] Consider an open, semi-circular arc of radius $r_o < d$ centered around the origin, as shown at right. Determine the line integral $\int_{L} \vec{B} \cdot d\vec{l}$. Explain your reasoning and/or show your work.

First of all, you don’t need to evaluate $\vec{B}$, but the integral $\int_{L} \vec{B} \cdot d\vec{l}$. So although the path is poorly constructed in that the direction is not with $\vec{B}$ nor is $\vec{B}$ constant along the path, that’s not important as you do not need to simplify the integral to pull $\vec{B}$ out.

That being said, the main problem is that Ampere’s Law as $\oint \vec{B} \cdot d\vec{l} = \mu_o \oint \vec{j} \cdot d\vec{a} = \mu_o I_{enc}$ can only be used for a loop that is closed.

So to fix this issue, one can close the loop! The two simplest closed loops involve adding a straight path from $R$ to the origin to $L$, or completing the circle clockwise. In the first case, the added path is in a region where $\vec{B} = 0$, with the given that $\vec{B}$ is zero at the origin with x-translational symmetry. In the second case the bottom part’s line integral will be equivalent to the top part’s line integral because of the discrete rotational symmetry about the $y$-axis through the origin.

So after closing the loop, one can find the current encircled (and divide by 2 if using the full circle).

Both methods will lead to a line integral of $-\frac{\mu_o J_{enc} \pi r_o^2}{2}$. Negative sign because the path is going in the “wrong” direction.

C. [8 pts] Consider the magnitude of the magnetic field for points inside the slab.

How does the magnitude of the magnetic field depend on $y$? Explain.

- **Constant in $y$**: Continuous translational symmetry along the $y$-axis.

How does the magnitude of the magnetic field depend on $z$? Explain.

- **Linear in $z$**: Looking at $\oint \vec{B} \cdot d\vec{l} = \mu_o \oint \vec{j} \cdot d\vec{a} = \mu_o I_{enc}$, moving the top part of a rectangular loop up by $\Delta z$ increases the current enclosed by a factor of $\Delta z$. $d\vec{l}$ is horizontal so it remains constant, which means the factor of $\Delta z$ must come from the magnitude of the magnetic field. You can also look at the Cartesian form of $\vec{\nabla} \times \vec{B} = \mu_o \vec{j}$ and argue that with symmetry arguments from part A and above (with the $x$-dependence constant as well), the only term is the $\frac{dB_z}{dz}$ term. Since this derivative is constant, the field must be linear in $z$. 