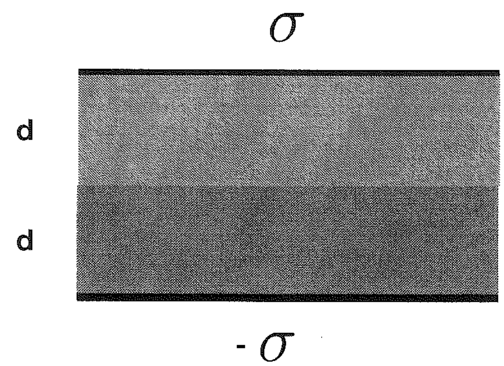
PHYSICS 321
CLASSICAL ELECTRODYNAMICS

27 Nov. 2019 Problem Set 6 These problems are due on Thursday, Dec. 6

1. *Parallel plate capacitor filled with dielectric* The space between the plates of a parallel-plate capacitor is filled with two slabs of linear dielectric material. Each slab has thickness d, so the total distance between the plates is $2d$. The upper slab has dielectric constant (ϵ_r) of 2, and the lower slab has a dielectric constant of 1.5. The free charge density on the top plate is σ and on the bottom plate it is $-\sigma$.



(a) Find the electric displacement $\mathbf{D}$ in each slab.

(b) Find the electric field $\mathbf{E}$ in each slab.

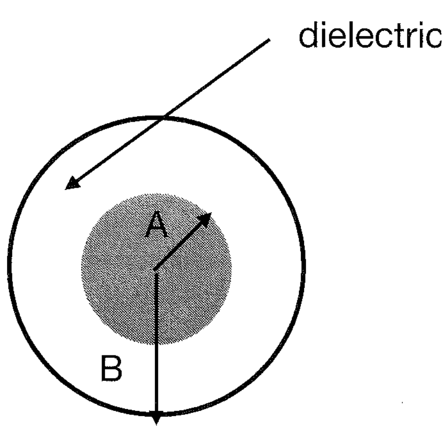
(c) Find the polarization $\mathbf{P}$ in each slab.

(d) Find the potential between the plates.

(e) Find the location and amount of all bound charge.

2. *Dielectric Shell* Find the electric field inside a dielectric shell of permitivity ϵ with inner radius A and outer radius B that is placed in a uniform electric field $\mathbf{E}_0$. Hint: use the potential in 3 regions and the boundary condition. The lecture notes have a similar problem.

3. *Cylindrical capacitor filled with dielectric*



(a) Find the capacitance per unit length of a coaxial cylindrical capacitor with inner radius A and outer radius B filled with a linear dielectric of permittivity ϵ.

(b) Find the capacitance per length if the dielectric fills only the lower half of the capacitor.

4. *Forces on Dielectrics* Two coaxial thin-walled conducting tubes with radii a and b are dipped vertically into a dielectric liquid of susceptibility χ_e and mass density ρ. If a voltage difference V_0 is applied to the tubes. The liquid rises to a. height h in the space between the tube walls.

(a) Determine h in terms of the parameters given above.

(b) Note that this is generally a small effect. For example, compute the value of h if the dielectric is water (see Table 4.2) for $a = 1.0$ cm, b=1.2cm, and $V_0 = 500$ volts.

1. a) $\vec{D}$ is ~~in~~ depends only on free charge density

$D = \sigma$ $\quad$ $\vec{D}$ points down
is the <u>same</u> in each slab

b)
$D = \epsilon E$ $\quad$ $\vec{E} \parallel \vec{D}$

upper slab: $\epsilon_r = 2$
$\epsilon = 2\epsilon_0$

$$|\vec{E}| = \frac{D}{2\epsilon_0} = \frac{\sigma}{2\epsilon_0}$$

lower slab $\quad \epsilon_r = 1.5$

$$E = \frac{\sigma}{1.5\,\epsilon_0}$$

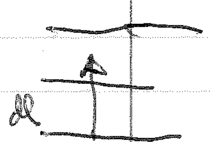
c) $\vec{P} = \vec{D} - \epsilon_0 \vec{E}$

upper slab $\quad P = \sigma - \epsilon_0 \dfrac{\sigma}{2\epsilon_0} = \dfrac{\sigma}{2}$

lower slab $\quad P = \sigma - \epsilon_0 \dfrac{\sigma}{1.5\epsilon_0} = \dfrac{1}{3}\sigma$

d) $V = -\displaystyle\int_{\text{lower}}^{\text{upper}} \vec{E}\cdot d\vec{l} = V(\text{upper}) - V(\text{lower})$

$\vec{E}$ points down and
$d\vec{l}$ points up



$$= +\int_0^d \frac{\sigma}{1.5\epsilon_0}\,dl - \int_d^{2d} \frac{\sigma}{2\epsilon_0}\,dl$$

$$= +\frac{\sigma d}{\epsilon_0}\left(\frac{1}{1.5} + \frac{1}{2}\right) = \frac{7}{6}\frac{\sigma}{\epsilon_0} d$$

e) P is constant in the seporate regions $\rho_b = 0$ everywhere except on boundaries

$\sigma_b = + P = \sigma$ at bottom upper slab
$= - \sigma/2$ at top of upper slab

$\sigma_b = + \sigma/3$ at bottom of lower slab
$= - \sigma/3$ at top of lower slab

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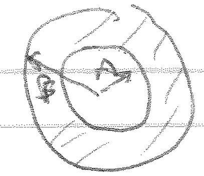
$\vec{E}_0$

z



Direction $\vec{E}_0$ define the z-axis
Origin at the center of spheres
At large distances

$$V \to -E_0 r\cos\theta = -E_0 z$$

$\cos\theta$ is $P_1(\cos\theta)$ thus we need only be concerned with terms with $l=1$

3 regions

Region I. $r < A$; II $A < r < B$; III $r > B$

ϵ_0 $\quad\quad$ ϵ $\quad\quad$ ϵ_0

$$V_I^{(r,\theta)} = a r\cos\theta$$

$$V_{II} = \left(c r + \frac{d}{r^2}\right)\cos\theta$$

$$V_{III}(r,\theta) = \frac{b}{r^2}\cos\theta - E_0 r\cos\theta$$

There are 4 unknowns a, b, c, d
Need 4 equations, must come from boundary conditions

continuity at $r = A$ $\quad$ $aA = cA + d/A^2$ (1)

" at $r = B$ $\quad$ $cB + d/B^2 = b/B^2 - E_0 B$ (2)

Boundary conditions on E involve ∇V

$$-\epsilon_0 \frac{\partial V_I}{\partial r}\bigg|_A + \epsilon \frac{\partial V_{II}}{\partial r}\bigg|_B = 0 \quad \text{No free charge}$$

also $\displaystyle -\epsilon \frac{\partial V_{II}}{\partial r}\Big|_B + \epsilon_0 \frac{\partial V_{III}}{\partial r}\Big|_B = 0$

Let $\epsilon/\epsilon_0 = \epsilon_r$

so have $\displaystyle \frac{\partial V_I}{\partial r}\Big|_A = \epsilon_r \frac{\partial V_{II}}{\partial r}\Big|_A$

$$a = \epsilon_r \left(c - \frac{2d}{A^3} \right) \qquad (3)$$

$$\epsilon_r \frac{\partial V_{II}}{\partial r}\Big|_B = \frac{\partial V_{III}}{\partial r}\Big|_B$$

$$\epsilon_r \left(c - \frac{2d}{B^3} \right) = \frac{-2b}{B^3} - E_0 \qquad (4)$$

Eqs (1)–(4) 4 eqns 4 unknowns Solve in Mathematica (next page)

We are asked for the $\vec{E}$ inside $r < A$

$$E_z = -\frac{\partial}{\partial z} a\, r\cos\theta = -a$$

$$= \frac{9\epsilon_r B^3 E_0}{(2\epsilon_r+1)(\epsilon_r+2)B^3 - 2A^3(\epsilon_r-1)^2}$$

Check $\epsilon_r \to 1$ $\qquad E_z \to \dfrac{9 B^3 E_0}{3 \cdot 3 \cdot B^3} = E_0$ ✓

Lim $A \to B$ $\qquad E_z \to \dfrac{9\epsilon_r E_0}{[2\epsilon_r^2 + 5\epsilon_r + 2 - 2(\epsilon_r^2 - 2\epsilon_r + 2)]}$

$= E_0$ ✓

3

a) In a coaxial cylindrical capacitor with inner radius A outer radius B, with dielectric ϵ. E is radial $\Rightarrow$ D is radial

The surface charge/area σ is free

$$D(A) = \sigma$$

$$D(s) = \sigma A / s$$

$$E(s) = D/\epsilon = \frac{\sigma A}{\epsilon s}$$

s is distance from center

Need potential diff V between conductors

$$V = \int_A^B ds \, \frac{\sigma A}{\epsilon s} = \frac{\sigma A}{\epsilon} \ln B/A$$

$$C = Q/V$$

$$Q = \sigma \, \text{area} = \sigma \, 2\pi A L \quad (\text{for a length } L)$$

$$C = \frac{\sigma 2\pi A L}{\frac{\sigma A}{\epsilon} \ln B/A} = \frac{\epsilon 2\pi L}{\ln B/A}$$

$$\frac{C}{L} = \frac{2\pi \epsilon}{\ln B/A}$$

b) Now the dielectric fills only the lower half of the capacitor. But $\vec{E}$ will still be radial. This is because there will be bound charge σ_B in addition to the free charge.

The total charge will be distributed evenly.

σ_B at

4. See Fig 4.32

Strategy find capacitance C as a function of h.

Then use $F = \frac{V_0^2}{2}\frac{dC}{dh}$

E is radial as before (Prob 3)

$$E(s) = E(a)\frac{A}{s}$$

$$V_0 = E(a)\, a \ln b/a \quad \text{as before}$$

Now get total charge (bound + free)

air $\sigma_{air} = \epsilon_0 E(a)$

water $\sigma_W = \epsilon E(a)$

Total charge $Q = \int \sigma_{air}\, da + \int \sigma_W\, da$ at $s=a$

$$= \epsilon_0 E_A 2\pi a(L-h) + \epsilon E_A 2\pi a(h)$$

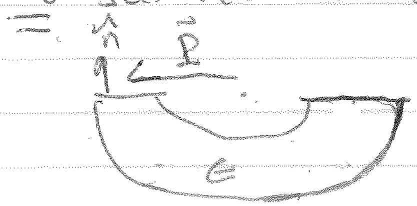
$$Q = E(a) 2\pi a\left[\epsilon_0(1+\chi_e)h + \epsilon_0(L-h)\right]$$

$$= E(a)(2\pi a)(\epsilon_0 \chi_e h + \epsilon_0 L)$$

$$= \frac{V_0}{a \ln b/a}(2\pi a)(\epsilon_0 \chi_e h + \epsilon_0 L)$$

A more mathematical explanation of the radial nature of $\vec{E}$:

If $\vec{E}$ is radial $\vec{P}$ will be radial so there is no surface charge (bound or free) on the filled surface



$\vec{P} \cdot \hat{n} = 0$

Tangential components of $\vec{E}$ are continuous a to the boundary so $\vec{E}$ can't change $\perp$ to its direction. Thus $\vec{E}$ will be independent of angle.

$$E(s) = \frac{E(A)\,A}{s} \Rightarrow V = E_A A \ln B/A \quad (1)$$

Total free charge Q = $\qquad$ Charge density $= \vec{E}\epsilon$

$$= \int_{below} \epsilon \vec{E} \cdot d\vec{a} + \int_{above} \epsilon_0 \vec{E} \cdot da$$

$$= E_A \, \epsilon(\pi A L) + E_A \epsilon_0 \pi A L$$

$$(2) \quad E_A = \frac{Q}{\pi L A(\epsilon+\epsilon_0)}$$

Using (2) in (1) $\quad V = \dfrac{Q}{\pi L(\epsilon+\epsilon_0)} \cancel{A} \ln B/A$

$$C = \frac{Q}{V} = \frac{(\epsilon_0+\epsilon)\pi L}{\ln B/A}$$

4. See Fig 4.32 λ = charge/unit length

Strategy find capacitance as a function of h then use $F = \frac{1}{2} V_0^2 \frac{dC}{dh}$

E is radial as in Prob 3

Air part $E = \frac{2\lambda}{4\pi\epsilon_0 s} \rightarrow V = \frac{2\lambda}{4\pi\epsilon_0} \ln(b/a)$

Liquid part $E = \frac{2\lambda}{4\pi\epsilon_0 s}$ same V

Compute the total charge Q $\lambda = \sigma\, 2\pi a$

~~For each~~ Air part $\sigma = \epsilon_0 E(a)$

water $\sigma = \epsilon E(a)$

$$Q = 2\pi a\,(\epsilon\, E(a)\, h + \epsilon_0\, E(a)\,(l-h))$$

$$= 2\pi a\,(\epsilon_0(\chi_e+1)\, \overset{E(a)}{h} + \epsilon_0\, E(a)\,(l-h))$$

$$= 2\pi a\, E(a)\left[\epsilon_0 \chi_e + \epsilon_0\, l\right]$$

$$C = \frac{Q}{V_0} = 2\pi \,(\ln b/a)^{-1} (\epsilon_0)(\chi_e h + L)$$

$$F_{up} = \frac{V_0^2}{2}\frac{dC}{dh} = \frac{V_0^2}{2} \frac{2\pi}{\ln(b/a)} \epsilon_0 \chi_e$$

This must over come the force of gravity F_g

$$F_g = mg = \rho\pi\,(b^2 - a^2)\, g\, h \qquad m = \rho(\text{Volum})$$

$$\text{so} \quad \boxed{h = \frac{V_0^2}{\ln b/a} \frac{\epsilon_0 \chi_e}{\rho\,(b^2 - a^2)\, g}}$$

(b) $h = \;= \dfrac{(500)^2}{\ln 1.2} \dfrac{8.85 \times 10^{-12}\ (79.1)}{(997)(10^{-4})(0.44)(9.8)}$

$= 0.0022 \ m$

$\epsilon_0 \chi_e = (\epsilon - \epsilon_0) = \epsilon_0 (80.1 - 1) = \epsilon_0\, 79.1$

$\rho = 997 \ kg/m^3$

$a = 10^{-2}\,m \quad b = 1.2 \times 10^{-2}\,m$

$g = 9.8 \ m/s^2$

Check units

$$[F] = V_0^2 \epsilon_0 = V_0 \epsilon_0 \ V_0 = \frac{q}{\text{distance}} \left[\frac{F\,\text{distance}}{q}\right]$$

✓