PHYSICS 321
CLASSICAL ELECTRODYNAMICS

7 Nov. 2019 Problem Set 6 These problems are due on Thursday , Nov. 14

1. *Uniform surface charge* Consider a circular disk of radius R and constant surface charge density σ. See Figure 2.34c on page 87. Set up a coordinate system in which there is azimuthal symmetry.

(a) Determine the potential $V(r,\theta)$ for positions with distances r from the center of the disk such that $r > R$.

(b) Determine the potential $V(r,\theta)$ for positions, not on the disk, with distances r from the center of the disk such that $r < R$.

2. *Cylindrical symmetry* A very, very long charged line (along the z-axis) with constant linear charge density λ is the simplest cylindrically symmetric system. The electric field can be obtained from the Gauss law and the potential V can be obtained by integration. For an infinitely long line charge $V(s) = -\frac{\lambda}{2\pi\epsilon_0}\ln s/s_0$, where the value of s_0 is arbitrary. This potential has the unrealistic property that it is infinite for $s = 0$ and for $s = \infty$. This is because the line is of infinite length and of zero width. However, the work required to take a unit charge from s_1 to s_2 is finite: $V(s_2) - V(s_1) = (\lambda/2\pi\epsilon_0)\ln s_1/s_2$.

(a) Check that the stated potential $V(s)$ yields the correct electric field, $\mathbf{E}$.

Here we'll find the same result for $V(s)$ by using the solution to Laplace's equation. There are two symmetries- translational invariance in z and rotational invariance in ϕ the azimuthal angle. Therefore the potential can depend only on s, the perpendicular distance to the line. In cylindrical coordinates Laplace's equation becomes $\frac{1}{ds}\frac{d}{s}\left(s\frac{dV}{ds}\right) = 0$ which has the general solution $V(s) = A\ln(s/s_0) + B$, where A and B are constants and s_0 is an arbitrary length.

(b) Determine A and B for the infinitely long line charge of constant λ.

Next consider a conducting cylinder of radius a which carries a constant line charge density λ spread uniformly on its surface

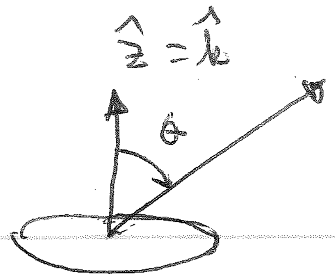
(c) Determine the potential for $s \geq a$.

(d) Consider two concentric conducting cylinders. The potential of the inner cylinder, of radius a, is set at V_0; the outer cylinder of radius b is grounded ($V = 0$). Determine the potential $V(s)$ and the electric field for $a \leq s \leq b$.

(e) Determine the capacitance per unit length.

3. *Force between two dipoles* Two dipoles, $\vec{p}_1$ and $\vec{p}_2$ are separated by a displacement $\vec{r}$. Determine the force between these objects. These are "pure" dipoles in the sense of Griffiths: their only relevant property is to have a dipole moment. Is your result consistent with Newton's third law?

PROBLEM #1



Preparation. To do this problem we need solutions of Laplace's equation and the potential along the z-axis. $\equiv V(r, \theta=0)$

$$\sigma = \frac{Q}{\pi R^2}$$

$$V(r, \theta=0) = \frac{1}{4\pi\epsilon_0} \int \frac{\sigma \, da'}{|\vec{r} - \vec{r}\,'|}$$

da' in cylindrical coord is $s\,ds\,d\phi \rightarrow 2\pi\, s\,ds$

$$|\vec{r} - \vec{r}\,'| = \sqrt{r^2 + r'^2 - 2\vec{r}\cdot\vec{r}\,'} \qquad \text{here } r' = s$$

$$\vec{r} = r\,(\pm\hat{k})$$

$$= \sqrt{r^2 + s^2} \qquad \vec{r}\cdot\vec{r}\,' = 0$$

$$V(r, \theta=0) = \frac{1}{4\pi\epsilon_0} \, 2\pi \frac{Q}{\pi R^2} \int_0^R \frac{s\,ds}{\sqrt{s^2 + r^2}}$$

$$= \frac{2Q}{4\pi\epsilon_0} \frac{1}{R^2} \left(\sqrt{s^2 + r^2} \right) \Bigg|_{s=0}^{s=R}$$

$$= \frac{2Q}{4\pi\epsilon_0} \frac{1}{R^2} \left(\sqrt{R^2 + r^2} - r \right) \qquad (1)$$

To proceed it is useful to have the

power series with $x \leq 1$

$$\sqrt{1+x^2} = 1 + \frac{1}{2}x^2 - \frac{1}{2}\frac{1}{4}x^4$$

$$+ \frac{1}{2}\frac{3}{4}\frac{1}{6}x^6 - \frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}\frac{1}{8}x^8 + \cdots$$

$$\equiv \sum_{n=0} C_n x^{2n} \quad \text{with } C_0 = 1,\ C_1 = 1/2, \text{etc} \qquad (1)$$

Now ready to study the separate cases $r < R$, $r > R$

a) $r > R$ Use (1)

$$V(r, \theta=0) = \frac{2Q}{4\pi\epsilon_0 R^2} r\left(\sqrt{1+\frac{R^2}{r^2}} - 1\right)$$

$$= \frac{2Q}{4\pi\epsilon_0 R^2} r \sum_{n=1} C_n \left(\frac{R}{r}\right)^{2n}$$

$$= \frac{2Q}{4\pi\epsilon_0} \sum_{n=1} C_n \frac{R^{2n-2}}{r^{2n-1}} \qquad (2)$$

The general solution for $r > R$ is $\displaystyle\sum_{\ell=0}^{\infty} \frac{B_\ell}{4\pi\epsilon_0} \frac{P_\ell(\cos\theta)}{r^{\ell+1}}$

So with $P_\ell(\theta=0, \cos\theta=1) = 1$

$$\frac{2Q}{\cancel{4\pi\epsilon_0}} \sum_{n=1} C_n \frac{R^{2n-2}}{r^{2n-1}} = \sum_{\ell=0} \frac{B_\ell}{\cancel{4\pi\epsilon_0}} \frac{1}{r^{\ell+1}}$$

$m = n-1$

$$= 2Q \sum_{m=0} C_{m+1} \frac{R^{2m}}{r^{2m+1}} = \sum_{\ell=0} \frac{B_\ell}{r^{\ell+1}}$$

matches if $\ell = 2m$

So $B_\ell = 2Q\, C_{\frac{\ell}{2}+1} R^\ell$ if ℓ = even

$= 0$ ℓ = odd

$r > R$

$$\text{So } V(r,\theta) = \frac{1}{4\pi\epsilon_0} 2Q \sum_{\ell=\text{even}} C_{\frac{\ell}{2}+1} \frac{R^\ell}{r^{\ell+1}} P_\ell(\cos\theta) \quad (3)$$

with the C's given on previous page

That only even P_ℓ enter is expected from the symmetry of the problem

b) The situation for $r < R$ is more subtle because even if $\vec{r}$ is not on the disk, there is an additional boundary condition on the disk

$$\boxed{E_z(\text{above}) - E_z(\text{below}) = \sigma/\epsilon_0}$$

This means we treat positions above the disk and below differently

Start with above $z > 0$ but $|z| < R$

Then use (i) in the form

$$V(r, \theta=0) = \frac{2Q}{4\pi\epsilon_0}\frac{1}{R^2} R\left(\sqrt{1+\frac{r^2}{R^2}} - \frac{r}{R}\right)$$

$$= \frac{2Q}{4\pi\epsilon_0}\frac{1}{R}\left[\sum_{n=0} C_n\left(\frac{r}{R}\right)^{2n} - \frac{r}{R}\right]$$

This must be matched to the general form

$$\frac{1}{4\pi\epsilon_0}\sum_{l=0}^{\infty}\left(A_l r^l + \frac{D_l}{r^{l+1}}\right) P_l(\cos\theta)$$

at $\theta = 0°$, $\cos\theta = 1$ $\rightarrow$ $D_l = 0$

and $A_l = \frac{2Q}{R} C_{l/2} R^{-l}$ $\quad l = \text{even}$

$= 0 \quad l = \text{odd}$

Thus for $0^\circ < \theta < 90^\circ$, $0 < \theta < \pi/2$

$$V(r,\theta) = \frac{1}{4\pi\epsilon_0} \sum_{\substack{l = \text{even} \\ = 0, \text{even}}} \frac{2Q}{R^{l+1}} C_{l/2}\, r^l P_l(\cos\theta)$$

$$- \frac{2Q}{4\pi\epsilon_0} \frac{r}{R^2} \cos\theta \qquad (4)$$

for $\pi > \theta > \pi/2$

$$V(r,\theta) = \frac{1}{4\pi\epsilon_0} \sum_{\substack{l=0, \\ \text{even}}} \frac{2Q}{R^{l+1}} C_{l/2}\, r^l P_l(\cos\theta)$$

$$+ \frac{2Q}{4\pi\epsilon_0} \frac{r}{R^2} \cos\theta \qquad (5)$$

in both cases $r\cos\theta = |z|$

(5) accounts for the Boundary Condition

and Laplace's equation

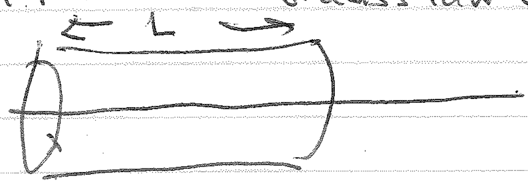
#2 a) $V(s) = \dfrac{-\lambda}{2\pi\epsilon_0} \ln s/s_0$

$$\vec{E} = -\vec{\nabla} V = -\hat{s}\frac{\partial}{\partial s}\left(\frac{-\lambda}{2\pi\epsilon_0} \ln s/s_0\right)$$

$$= \hat{s}\frac{\lambda}{2\pi\epsilon_0}\frac{1}{s}$$

This is the result from Gauss Law

with a Gaussian cylinder:



$$EA = \frac{\text{charge enclosed}}{\epsilon_0} = \lambda L$$

$$A = 2\pi s L$$

$$E = \frac{\lambda L}{2\pi s \epsilon_0 L} = \frac{\lambda}{2\pi\epsilon_0}\frac{1}{s}$$

b) Laplace's equation is cylindrical

$$\frac{1}{s}\frac{d}{ds}s\frac{d}{ds}V(s) = 0$$

From symmetry $V(\vec{s}) = V(s)$

so $s\dfrac{dV}{ds} = \text{const}$

$$\therefore \frac{dV}{ds} = \frac{\text{const}}{s} \quad ; \; V(s) = A \ln s/s_0 + B$$

We may set $B = 0$ because it is a constant

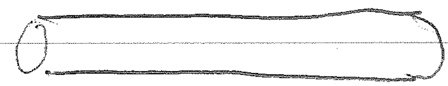
The boundary condition on V must be

obtained from Gauss Law

So compute $\vec{E} = -\vec{\nabla} V$
and use Gauss Law to determine

$$A = \frac{-\lambda}{2\pi\epsilon_0}$$

s_0 is a constant that is undetermined

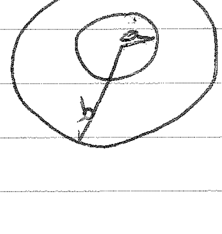
c) now we have 

$\lambda = \sigma 2\pi a$, $\sigma = \lambda/(2\pi a)$ surface charge density

again Gauss Law says

$$V = \frac{-\lambda}{2\pi\epsilon_0} \ln s/a$$

where can pick $s_0 = a$ to
arbitrarily make $V = 0$ on the surface



$$V(s) = A \ln s/s_0 \cancel{+B}$$

$$V(s=a) = V_0 = A \ln a/s_0 \quad (1)$$
$$V(s=b) = 0 = A \ln b/s_0 \quad (2)$$

From (2) $s_0 = b$

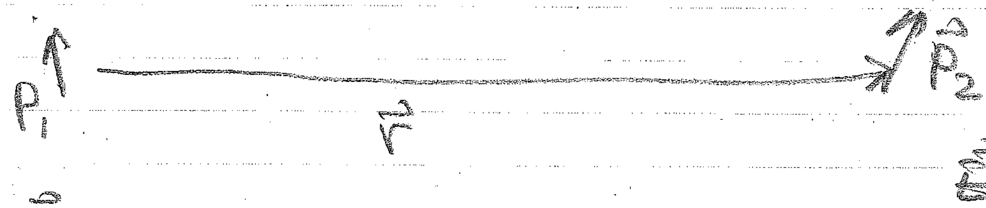
From (1) $V_0 = A \ln a/b$

$$A = V_0/\ln(a/b)$$

$$V(s) = \frac{V_0}{\ln(a/b)} \ln(s/b)$$

PROBLEM #3

Force between two dipoles



$\vec p_1$ creates an electric field which acts on $\vec p_2$. $\vec E_1$ is given by

$$\vec E_1 = \frac{1}{4\pi\epsilon_0} \frac{3\,\vec p_1\cdot\hat r\,\hat r - \vec p_1}{r^3}$$

. This field varies over the extent of the dipole 2. According to Eq (4.5) of the text, the force on dipole 2 is given by

$$\vec F = \vec p_2\cdot\vec\nabla\,\vec E_1 = \vec p_2\cdot\vec\nabla\left[\frac{1}{4\pi\epsilon_0}\left(\frac{3\,\vec p_1\cdot\vec r\,\vec r}{r^5} - \frac{\vec p_1}{r^3}\right)\right]$$

$$\left(\vec p_2\cdot\vec\nabla\right)\left(\vec p_1\cdot\vec r\right)\vec r = \vec p_2\cdot\vec p_1\,\vec r + \vec p_1\cdot\vec r\,\vec p_2 \qquad (1)$$

$$\vec p_2\cdot\vec\nabla\,\frac{1}{r^5} = -\frac{5}{r^6}\,\vec p_2\cdot\hat r$$

$$\vec p_2\cdot\vec\nabla\,\frac{1}{r^3} = -\frac{3}{r^4}\,\vec p_2\cdot\hat r$$

Use these terms in (1) to get

$$\vec{F}(1 \text{ on } 2) = \frac{1}{4\pi\epsilon_0}\left[\frac{3\,\vec{P}_2\cdot\vec{P}_1\,\vec{r} + 3\vec{P}_1\cdot\vec{r}\;\vec{P}_2}{r^5}\right.$$

$$\left. - \frac{15}{r^6}\;\vec{P}_1\cdot\vec{r}\;\vec{r}\cdot\vec{P}_2\,\hat{r} + \frac{3\,\vec{P}_1\,(\vec{P}_2\cdot\hat{r})}{r^4}\right]$$

$$= \frac{1}{4\pi\epsilon_0}\left[\frac{3(\vec{P}_1\cdot\vec{P}_2\,\hat{r} + \vec{P}_1\cdot\hat{r}\,\vec{P}_2)}{r^4} - \frac{15\,\vec{P}_1\cdot\hat{r}\,\vec{P}_2\cdot\hat{r}\,\hat{r}}{r^4}\right.$$

$$\left. + \frac{3\,\vec{P}_1\;(\vec{P}_2\cdot\hat{r})}{r^4}\right]$$

$$= \frac{1}{4\pi\epsilon_0}\left[3\,\frac{\vec{P}_1\cdot\vec{P}_2\,\hat{r}}{r^4} + 3\,\frac{(\vec{P}_1\cdot\hat{r}\,\vec{P}_2 + \vec{P}_2\cdot\hat{r}\,\vec{P}_1)}{r^4}\right.$$

$$\left. - \frac{15\,\vec{P}_1\cdot\hat{r}\,\vec{P}_2\cdot\hat{r}\,\hat{r}}{r^4}\right]$$

To get $\vec{F}(2 \text{ on } 1)$ interchange $\vec{P}_1$ and $\vec{P}_2$ and $\vec{r} \Rightarrow -\vec{r}$ $(\hat{r} \to -\hat{r})$. Thus

$\vec{F}(1 \text{ on } 2) = -\vec{F}(2 \text{ on } 1)$ N's 3rd Law.