PHYSICS 321
CLASSICAL ELECTRODYNAMICS

30 Oct. 2019 Problem Set 5 These problems are due on Thursday , Nov. 7

1. *Potential on a sphere* The potential on the surface of a sphere of radius R is given by $V(\theta) = V_0 \cos 4\theta$. There is no charge inside or outside the sphere.

(a) Determine the potential $V(r, \theta)$ for positions inside and outside the sphere.

(b) Determine the surface charge density $\sigma(\theta)$ on the sphere.

2. *Image charge and capacitance* Consider a long wire of radius R with charge per unit length λ. The wire is suspended a distance $d \gg R$ above a very large grounded plane that can be approximated as infinitely large. Determine the capacitance per unit length.

3. *Multipole expansion* Three charges are arranged in a linear array. A charge $-2q$ is placed at the origin, and two charges, each of $+q$ are placed at $(0, 0, L)$ and $(0, 0, -L)$. Consider the multipole expansion

$$V(r, \theta) = \frac{1}{4\pi\epsilon_0} \sum_{l=0}^{\infty} \frac{B_l}{r^{l+1}} P_l(\cos\theta)$$

$$B_l \equiv \int d^3r' \rho(\mathbf{r}') r'^l P_l(\cos\theta') = \sum_n q_n r_n^l P_l(\cos\theta_n),$$

with the second expression for B_l applying for the case of point charges.

(a) Determine the multipole coefficients that are non-vanishing.

(b) Compute the non-vanishing B_l for the two lowest values of l.

4. *Boundary value problem* The potential on the surface of a sphere of radius R is given by:

$$V(R, \theta) = V_0,\ 0^\circ \leq \theta \leq 90^\circ; \quad V(R, \theta) = -V_0,\ 90^\circ \leq \theta \leq 180^\circ$$

(a) Find the potential $V(r, \theta)$ outside and inside the sphere. You may express your answer in terms of a well-defined integral of a Legendre polynomial, $I_n = \int_0^1 dx P_n(x)$.

(b) Find the surface charge density on the sphere.

(c) Find the electric dipole and octupole moments of the sphere by (i) identifying them from the Legendre polynomial expansion of the potential. (ii) integrating over the surface charge density.

1 Inside the sphere

a) $V_{in}(r,\theta) = V_0 \sum_{n=0}^{\infty} \frac{r^n}{R^n} P_n(\cos\theta)\, a_n \qquad r < R \qquad (1)$

Outside

$V_{out}(r,\theta) = V_0 \sum_{n=0}^{\infty} \frac{R^{n+1}}{r^{n+1}} P_n(\cos\theta)\, a_n \qquad r > R \qquad (2)$

Continuity and proper dimensions are incorporated above

The unknown is a_n, to be determined from the given

$V(R,\theta) = V_0 \cos 4\theta$

We need to expand $\cos 4\theta$ in P_n

$\cos 4\theta = \cos 4(\pi-\theta)$ so n is even

it's highest value is 4 $\left(e^{4i\theta} = (e^{i\theta})^4 \right)$

$$\cos 4\theta = \sum_{n=0,2,4} a_n P_n(\cos\theta)$$

Integrals done with Mathematica

$$a_n = \frac{2n+1}{2} \int_0^{\pi} \sin\theta\, d\theta\, P_n(\cos\theta) \cos 4\theta$$

$$a_0 = +\frac{1}{2} \left(-\frac{2}{15}\right) = -\frac{1}{15}$$

$n = 0$, $P_0(\cos\theta) = 1$

$$a_2 = \frac{5}{2} \int_0^{\pi} \sin\theta\, d\theta\, P_2(\cos\theta) \cos 4\theta$$

$$= -\frac{5}{2} \frac{(32)}{105} = \frac{-16}{21}$$

$$a_4 = \frac{9}{2} \int_0^{\pi} \sin\theta\, d\theta\, P_4(\cos\theta) \cos 4\theta = \frac{9}{2} \cdot \frac{128}{315} = \frac{64}{35}$$

Thus $\cos 4\theta = -\frac{1}{15} P_0(\cos\theta) - \frac{16}{21} P_2(\cos\theta) + \frac{64}{35} P_4(\cos\theta)$

$\boxed{a_0 = -\frac{1}{15}, \quad a_2 = \frac{-16}{21}, \quad a_4 = \frac{64}{35}}$ This along with eqs(1,2) gives the $V(r,\theta)$.

(b) From Eq (2.31) we have

$$\vec{E}_{out} \cdot \hat{r} - \vec{E}_{in} \cdot \hat{r} = \frac{\sigma(\theta)}{\epsilon_0}$$

$$-\left.\frac{\partial V_{out}}{\partial r}\right|_{r=R^+} + \left.\frac{\partial V_{in}}{\partial r}\right|_{r=R^-} = \frac{\sigma(\theta)}{\epsilon_0} \qquad \text{Use (1,2)}$$

$$V_0 \sum_{n=0,2,4} \left(a_n P_n(\cos\theta) \frac{R^{n+1}}{R^{n+2}} (n+1) + a_n P_n(\cos\theta)\, n \frac{R^{n-1}}{R^n} \right) = \frac{\sigma(\theta)}{\epsilon_0}$$

$$\sigma(\theta) = \frac{\epsilon_0 V_0}{R} \sum_{n=0,2,4} (2n+1) a_n P_n(\cos\theta) =$$

$$= \frac{\epsilon_0 V_0}{R} \left[\frac{-1}{15} - \frac{5 \cdot 16}{21} P_2(x) + \frac{9 \times 64}{35} P_4(x) \right]$$

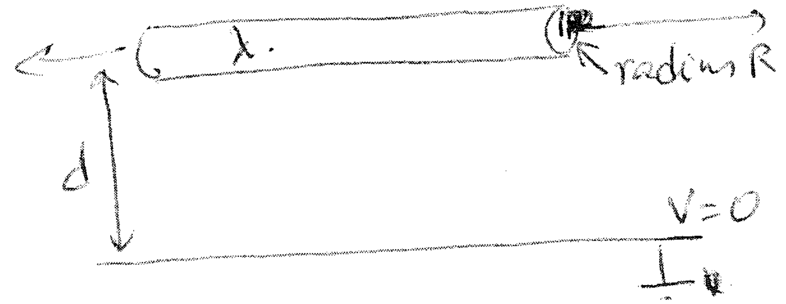
$$= \frac{\epsilon_0 V_0}{R} \left[-\frac{1}{15} - \frac{40}{21} (3x^2 - 1) + \frac{72}{35} (3 - 30x^2 + 35x^4) \right]$$

$$= \frac{\epsilon_0 V_0}{R} \left[\frac{841}{105} - \frac{472}{7} x^2 + 72 x^4 \right]$$

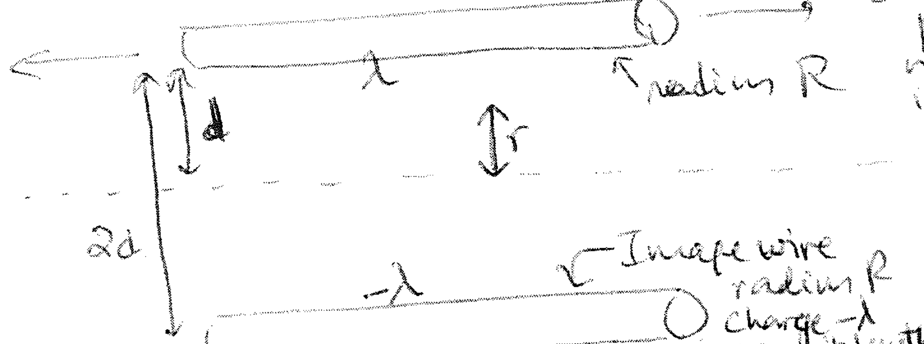
$x = \cos\theta$

P321 Question #2

Solns to HW#4

3]. 

Solve the image problem: Imagine an image wire with charge density $-\lambda$ at position $-d$. $V=0$ at mid plane. Capacitance is that of these 2 wires.



To find the capacitance, first find the potential difference ΔV between the wire and its image.

then C per unit length $= \dfrac{\lambda}{\Delta V}$

Use Gauss' Law on a Gaussian cylinder radius $r>R$; length L

$$E(r) \cdot 2\pi r L = \frac{\lambda L}{\epsilon_0} \Rightarrow E(r) = \frac{\lambda}{2\pi\epsilon_0} \cdot \frac{1}{r}$$

and $V(r) = -\int_P^r E(r)\,dr$ P is the chosen point for $V=0$

$$= -\frac{\lambda}{2\pi\epsilon_0} \ln r + C$$ where C is a constant

Potential at position r above plane (see figure).

$$\frac{+\lambda}{2\pi\epsilon_0}\ln(r+d) - \frac{\lambda}{2\pi\epsilon_0}\ln(d-r) = \frac{\lambda}{2\pi\epsilon_0}\ln\frac{d+r}{d-r}$$

$$-\frac{\lambda}{2\pi\epsilon_0}\ln d \qquad + \frac{\lambda}{2\pi\epsilon_0}\ln d$$

(Image Wire) (Original Wire)

3] (cont'd)

$$\Delta V = \frac{\lambda}{2\pi\epsilon_0}\left[\ln\frac{d+(d-R)}{d-(d-R)} - \ln\frac{d+(-d+R)}{d-(-d+R)}\right]$$

$$= \frac{\lambda}{2\pi\epsilon_0}\left[\ln\frac{2d-R}{R} - \ln\frac{R}{2d-R}\right]$$

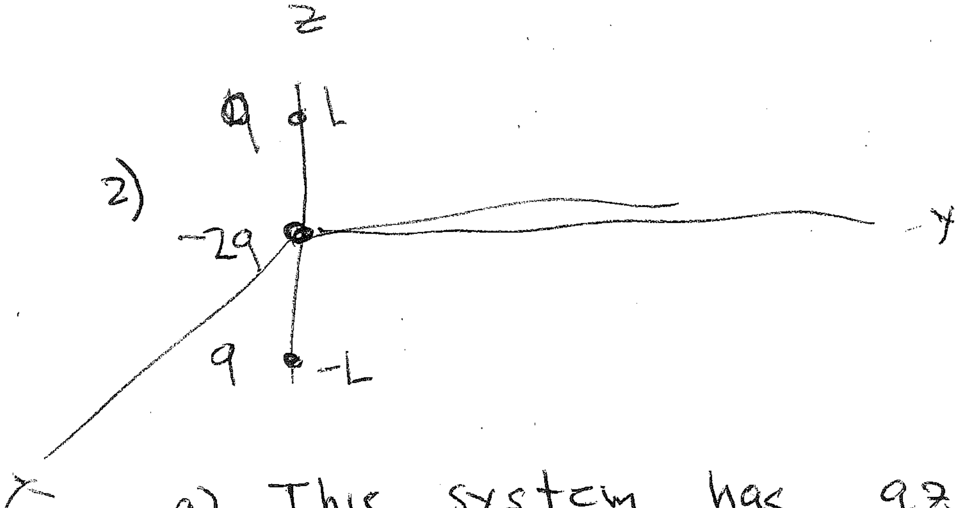
$$\Delta V = \frac{\lambda}{2\pi\epsilon_0}\, 2\ln\left(\frac{2d-R}{R}\right) \approx \frac{\lambda}{\pi\epsilon_0}\ln\frac{2d}{R}$$

since $d \gg R$.

and capacitance per unit length:

$$C = \frac{\lambda}{\Delta V} = \frac{\lambda}{\frac{\lambda}{\pi\epsilon_0}\ln\frac{2d}{R}} = \frac{\pi\epsilon_0}{\ln(2d/R)}$$

2)



a) This system has azimuthal symmetry and another symmetry. If $\theta \to \pi - \theta$ there is no change. This is top vs down symmetry so l must be even. The total charge $= 0$, so

l is even and $l > 0$

b) we need B_2, B_4

as a check do B_0, B_1. Let $q_1 = q, \theta_1 = 0$, $q_2 = -2q, \theta_2 = \frac{\pi}{2}$, $q_3 = q, \theta_3 = \pi$

$$B_0 = \sum_l q_n r_n^l \Big|_{l=0} = \sum_n q_n = q - 2q + q = 0$$

$$B_1 = Lq\, \underbrace{P_1(\cos\theta)}_{=1} - 2q \overset{0}{\cdot 0}\, P_1(\pi/2) + qL\, P_1(\pi)$$

$$= Lq - qL = 0$$

$$B_2 = qL^2 P_2(1) - 2q(0) P_2(0) + qL^2 \underbrace{P_2(-1)}_{=1}$$

$$= 2qL^2$$

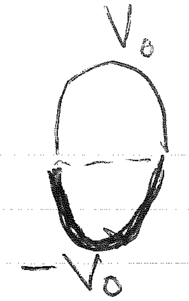
$$B_4 = qL^4 P_4(1) + 0 + qL^4 \underbrace{P_4(-1)}_{=1}$$

$$= 2qL^4$$

Thus $V(r,\theta) \approx \dfrac{2qL^2}{r^3} P_2(\cos\theta) + \dfrac{2qL^4}{r^5} P_4(\cos\theta)$

$P_2(x) = \dfrac{3x^2 - 1}{2}$, $P_4(x) = \frac{1}{8}(3 - 30x^2 + 35x^4)$

4.



Outside $V(r,\theta) = \sum_{l=0}^{\infty} \frac{b_l}{r^{l+1}} P_l(\cos\theta)$

Inside $V(r,\theta) = \sum_{l=0}^{\infty} a_l r^l P_l(\cos\theta)$

a)

The charge distribution is odd under the reflection $\theta \to \pi - \theta$ or $\cos\theta \to -\cos\theta$

Thus l is an odd integer

Conside $r \to R$ from outside.

In general

$$\int_{-1}^{1} d\cos\theta \, V(R,\theta) P_n(\cos\theta) = \frac{b_n}{R^{n+1}} \frac{2}{2n+1}$$

$$= V_0 \left(\int_0^1 dx \, P_n(x) - \int_{-1}^{0} dx \, P_n(x) \right)$$

In second integral change x to $-x$

$$= V_0 \int_0^1 dx \, \left(P_n(x) \, (1 - (-1)^n \right)$$

$$= 2V_0 \int_0^1 dx \, P_n(x) \quad \text{if } n = \text{odd}$$

$$= 0 \quad \text{if } n = \text{even}$$ as expected

Define $I_n = \int_0^1 dx \, P_n(x)$

Then $b_n = V_0 I_n (2n+1)$

$r > R$ $\quad V_{out}(r,\theta) = \sum_{n=odd} \frac{V_0 I_n (2n+1)}{r^{n+1}} R^{n+1} P_n(\cos\theta)$ (I)

by continuity $r < R$ $\quad V_{in}(r,\theta) = \sum_{n=odd} V_0 I_n (2n+1) \frac{r^n}{R^n} P_n(\cos\theta)$

b) $\sigma = $ -

$$\frac{\sigma}{\epsilon_0} = \hat{r}\cdot\vec{E}_{out} - \hat{r}\cdot\vec{E}_{in}$$

$$= -\frac{\partial V_{out}}{\partial r}(R,\theta) + \frac{\partial V_{in}}{\partial r}(R,\theta)$$

$$\sigma(\theta) = \epsilon_0 \left(\sum_{n=odd} \frac{V_0 I_n (2n+1)(n+1)}{R^{n+2}} P_n(\cos\theta) R^{n+1} \right.$$

$$- \sum_{n=odd} V_0\, n\, I_n (2n+1) \frac{R^{n-1}}{R^n} P_n(\cos\theta)$$

$$= \frac{V_0 \epsilon_0}{R} \sum_{n=odd} (2n+1) P_n(\cos\theta) I_n$$

c) Multipole expansion

$$V(\vec{r}) = \frac{1}{4\pi\epsilon_0} \sum_n \frac{1}{r^{n+1}} q_n P_n(\cos\theta) \qquad \text{II}$$

with q_n = multipole moments

$$= \int r^n P_n(\cos\theta) \rho(\vec{r})\, d\tau$$

(i)

Compare I & II

$$q_n = 4\pi\epsilon_0 \, V_0 \, I_n (2n+1) \, R^{n+1}$$

$$q_1 = 4\pi\epsilon_0 \, V_0 \, 3 I_1 \, R^2$$

$$q_3 = 4\pi\epsilon_0 \, V_0 \, 5 I_3 \, R^4$$

(ii)

We are asked for the dipole $l=1$ and octupole $l=3$ moments

For a surface charge on a sphere radius R

$$q_n = \int da\ \sigma(\theta)\ P_n(\cos\theta)\ R^n\ (2n+1)$$

$$= 2\pi \int_{-1}^{1} dx\ \sigma(\theta)\ P_n(\cos\theta)\ R^{n+2}\ (2n+1)$$

$$= \left(\frac{R^{n+2}}{R}\right) 2\pi\epsilon_0 \int_{-1}^{1} dx \sum_{m=odd} V_0\ (2m+1)\ P_m(\cos\theta)\ I_m\ P_n(\cos\theta)$$

$$= \left(\frac{R^{n+2}}{R}\right) 2\pi\epsilon_0\ V_0\ (2n+1)^{\cancel{2}}\ \frac{2}{\cancel{2n+1}}\ I_n$$

$$q_1 = R^2\ 4\pi\epsilon_0\ V_0\ I_1\ 3 \qquad I_1 = 1$$

$$q_3 = R^4\ 4\pi\epsilon_0\ V_0\ I_3\ 5, \quad I_3 = -1/8$$