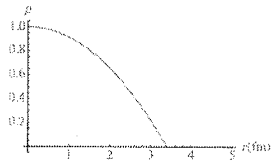
PHYSICS 321
CLASSICAL ELECTRODYNAMICS

8 Oct. 2019 Problem Set 3 These problems are due on Thursday , Oct 17

1. *Electric Field of an Atomic Nucleus* The radial dependence of the electric charge density inside a certain atomic nucleus of radius a is roughly described by the piecewise function: $\rho(r) = \rho_0(1 - r^2/a^2)$ if $r \leq a$, and $\rho = 0$ if $r > a$. You are given that $a = 3.4$ femtometers $=3.4$ fm.



(a) The nucleus contains 21 protons, Determine the value of r_0.

(b) Find $\mathbf{E}$ and V for positions outside the nucleus. What are their values at the surface?

(c) Find $\mathbf{E}$ and V for positions inside the nucleus. Determine their values at the center.

(d) In units of a, what is the radial location of the maximum magnitude of the electric field

(e) Plot E and V as functions of r/a for values of r/a between 0 and 5.

2. A thin rod of length L has its left end at $x = -L/2$ and its right end at $x = L/2$. The rod carries a line charge density given by $\lambda = \lambda_0 \frac{x^2}{L^2}$.

(a) Determine the electric field at the origin.

(b) Determine the electric potential V at all points in space. You can express your answer in terms of a well-defined one-dimensional integral.

3. Consider an infinitely long cylinder of radius a, with a uniform (constant) charge density, ρ. Determine the electric field (per unit length) for positions inside and outside the cylinder.

4 . *The potential energy of a sphere of charge*

(a) Calculate the electric potential energy of a sphere of radius R carrying a total charge Q uniformly distributed throughout its volume.

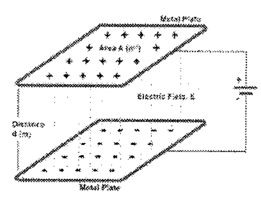
(b) Calculate the gravitational potential energy of a sphere of uniform density with radius R' and total mass M.

(c) Calculate the gravitational potential energy of the moon.

(d) Imagine that you can assemble a sphere of protons with a density equal to that of water. What would be the radius of this sphere if its electric potential energy were sufficient to blow up the moon?

(e) Determine the voltage at the surface of the sphere of protons?

5.



Two metal plates having and area A and separation d form a parallel plate capacitor. Take the area to be a square of length $L = \sqrt{A}$ with $L \gg d$. Let the vertical direction be the z axis and the horizontal direction the x axis. The potential at the top $z = d$ is held at a potential V_0, and that at the bottom is grounded (its potential is 0.

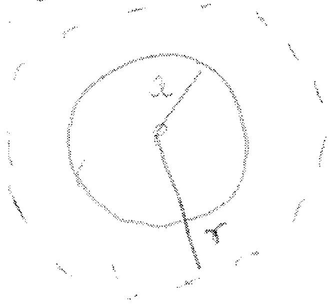
(a) Use Laplace's equation to determine the potential in the region between the plates.

(b) Determine $\mathbf{E}$.

(c) Determine the charge distribution on each plate.

(d) Determine the capacitance of the parallel-plate capacitor.

4] (b) cont'd.



By symmetry: $\vec{E}(\vec{r}) = E(r)\hat{r}$.

$$\oint_{\substack{\text{surface}\\ \text{of sphere}}} \vec{E}\cdot d\vec{A} = \frac{Q_{enc}}{\epsilon_0}$$ [Gauss' Law Integral form].

$$E(r)4\pi r^2 = \frac{1}{\epsilon_0}\int_0^a [\rho(r')4\pi r'^2]dr'$$ $\rho = 0$ ~~as all~~ outside nucleus.

$$= \frac{1}{\epsilon_0}\cdot 21e$$

$\Rightarrow$ $$\boxed{E(r) = \frac{1}{4\pi\epsilon_0}\cdot\frac{21e}{r^2}}$$ $r > a$. $\vec{E}(\vec{r}) = E(r)\hat{r}$.

$$V(r) = -\int_\infty^r E(r')dr' = -\frac{1}{4\pi\epsilon_0}\cdot 21e\int_\infty^r \frac{dr'}{r'^2} = -\frac{21e}{4\pi\epsilon_0}\left(-\frac{1}{r'}\right)\Bigg|_\infty^r$$

$$\boxed{V(r) = \frac{21e}{4\pi\epsilon_0}\cdot\frac{1}{r}}$$ for $r > a$.

At surface $r = a$, $$\boxed{\begin{aligned}\vec{E}(a) &= \frac{1}{4\pi\epsilon_0}\cdot\frac{21e}{a^2}\hat{r}\\ V(a) &= \frac{21e}{4\pi\epsilon_0 a}\end{aligned}}$$

(c) Inside nucleus, use same approach as in (b) but with sphere radius $r < a$

$$E(r)4\pi r^2 = \frac{1}{\epsilon_0}\int_0^{\cancel{a}\,r} \rho(r')4\pi r'^2 dr'$$

Q (c) cont'd.

$$E(r)\,4\pi r^2 = \frac{1}{\epsilon_0}\cdot \rho_0 4\pi\left[\frac{1}{3}r^3 - \frac{1}{5}\frac{r^5}{a^2}\right].$$

$$= \frac{\rho_0 4\pi}{\epsilon_0}\, r^3\left[\frac{1}{3} - \frac{1}{5}\left(\frac{r}{a}\right)^2\right]$$

$$\Rightarrow \boxed{E(r) = \frac{\rho_0 r}{\epsilon_0}\left[\frac{1}{3} - \frac{1}{5}\left(\frac{r}{a}\right)^2\right]}$$ for $r<a$

& $\vec{E}(\vec{r}) = E(r)\,\hat{r}$

and $$V(r) = -\int_\infty^r E(r')\,dr'$$

$$= -\int_\infty^a E(r')\,dr' - \int_a^r E(r')\,dr'$$

$$= \frac{Ze}{4\pi\epsilon_0 a} - \frac{\rho_0}{\epsilon_0}\left[\frac{1}{3}\int_a^r r'\,dr' - \frac{1}{5a^2}\int_a^r r'^3\,dr'\right]$$

$$\boxed{V(r) = \frac{Ze}{4\pi\epsilon_0 a} - \frac{\rho_0}{\epsilon_0}\left[\frac{1}{6}\left(r^2-a^2\right) - \frac{1}{20a^2}\left(r^4-a^4\right)\right]}$$ for $r<a$

Q 2(d) At center of nucleus, $r=0$

then $\boxed{E(0) = 0}$

$$V(0) = \frac{Ze}{4\pi\epsilon_0 a} - \frac{\rho_0}{\epsilon_0}\left[-\frac{a^2}{6} + \frac{a^4}{20a^2}\right]$$

from (a): $Ze = 4\pi\rho_0\frac{2}{15}a^3$

$$= \frac{\rho_0}{\epsilon_0}\frac{2}{15}a^2 + \frac{\rho_0 a^2}{\epsilon_0}\left[\frac{1}{6} - \frac{1}{20}\right].$$

$$\boxed{V(0) = \frac{\rho_0}{\epsilon_0}\,a^2\,\frac{1}{4}}$$

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Solns to HW #2

4] (d) Maximum of $E(r)$ where (here $\vec{E}(\vec{r}) = E(r)\hat{r}$).

$$E(r) = \begin{cases} \frac{1}{4\pi\epsilon_0} \cdot \frac{2 Q_{ic}}{r^2} = \frac{\rho_0}{\epsilon_0} \frac{2}{15} \frac{a^3}{r^2} & (r > a) \\ \frac{\rho_0 r}{\epsilon_0}\left[\frac{1}{3} - \frac{1}{5}\left(\frac{r}{a}\right)^2\right] & (0 < r < a) \end{cases}$$

fields drops as $\frac{1}{r^2}$ for $r > a$, so max. must be in the region $0 < r \leq a$.

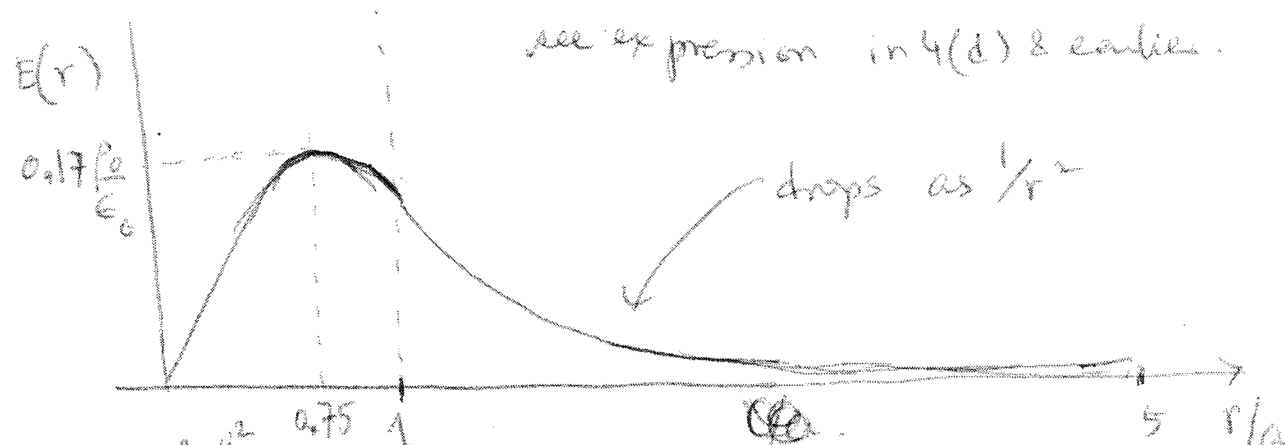
Take usual approach & set $\frac{\partial E}{\partial r} = 0$ to get:

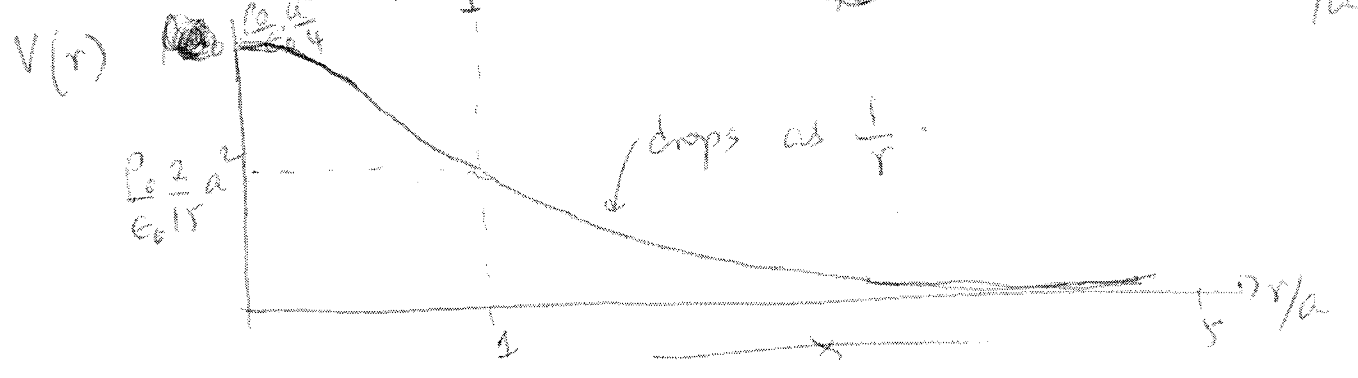
$$\left[\frac{1}{3} - \frac{1}{5}\frac{r^2}{a^2}\right] + r\left[-\frac{2}{5}\frac{r}{a^2}\right] = 0$$

$$\Rightarrow \frac{3}{5}\left(\frac{r}{a}\right)^2 = \frac{1}{3} \Rightarrow \boxed{\frac{r}{a} = \sqrt{\frac{5}{9}} = \frac{\sqrt{5}}{3}} \approx 0.75$$

By inspection, $\left.\frac{\partial^2 E}{\partial r^2}\right|_{r=\frac{\sqrt{5}}{3}a} < 0$, therefore $\frac{r}{a} = \frac{\sqrt{5}}{3}$ is the radial location of maximum electric field magnitude.

(e). see expression in 4(d) & earlier.





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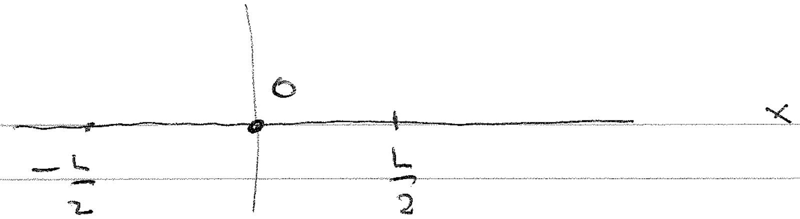


$\lambda(x) = \frac{x^2}{L^2}$

a) Determine $\vec{E}(\vec{x}=0)$. Intuitively $\boxed{\vec{E}(\vec{x}=0) = \vec{0}}$ because contributions from left & right cancel. From the calculation

$$\vec{E}(\vec{x}) = \frac{1}{4\pi\epsilon_0} \int_{-L/2}^{L/2} dx' \frac{\lambda(x')\,(\vec{x}-\vec{x}')}{|\vec{x}-\vec{x}'|^3}$$

at $\vec{x} = 0$

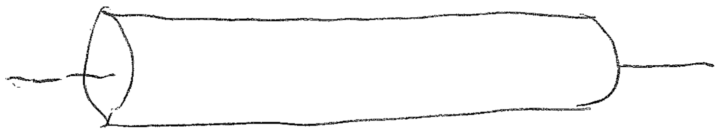
$$\vec{E}(0) = \frac{1}{4\pi\epsilon_0} \frac{(-)\lambda_0}{L^2} \int_{-\frac{L}{2}}^{L/2} dx' \frac{x'^2 \, x'}{|\vec{x}'|^3} \hat{i}$$

$$= \frac{-\lambda_0}{4\pi\epsilon_0} \frac{1}{L^2} \int_{-\frac{L}{2}}^{L/2} dx' \frac{x'}{|\vec{x}'|}$$

$$= -\frac{\lambda}{4\pi\epsilon_0} \frac{1}{L^2} \left[\int_{-L/2}^{0} dx'(-1) + \int_{0}^{L/2} dx' \right]$$

$$= -\frac{\lambda}{4\pi\epsilon_0} \frac{1}{L^2} \left(-L/2 + \frac{L}{2} \right) = 0$$

b) $$V(\vec{x}) = \frac{1}{4\pi\epsilon_0} \frac{\lambda_0}{L^2} \int_{-\frac{L}{2}}^{L/2} dx' \frac{x'^2}{|\vec{x}-\vec{x}'|^2}$$

3. 

Infinitely long cylinder. We have cylindrical symmetry. $\vec{E}$ points in the radial direction

Use Gauss Law with cylindrical symmetry

take a cylinder of Length L radius s

$$\oint \vec{E}\cdot d\vec{a} = \frac{Q\,(\text{enclosed})}{\epsilon_0}$$

$\vec{E} = E(r)\,\hat{r}$ where $\hat{r}$ point outward

Inside front cover cyl. coordinates use $\vec{s}$ (not $\vec{r}$)

so $\vec{E}(s) = E(s)\,\hat{s}$, $d\vec{a} = da\,\hat{s}$

$\oint \vec{E}\cdot d\vec{a} = ?$ E is constant on surface of cylinder

$d\vec{a} = da\,\hat{s}$

Thus $\oint \vec{E}\cdot d\vec{a} = E(s)\int da = E(s)\,L\,2\pi s$

$Q\,(\text{enclosed}) = \int \rho(\vec{r}\,')\,d^3r'$

if $s < a$, $Q(\text{enc}) = \int \rho(\vec{r}\,')\,d^3r' = \rho\int_0^s 2\pi\, s'\,ds'\,L$

$Q(\text{enc}) = \rho\, 2\pi L\,\frac{s^2}{2} = \rho\pi L s^2$

so $s<a$ $\quad E(s)\,2\pi L s = \frac{\rho}{\epsilon_0}\pi L s^2$

$$\boxed{E(s) = \frac{\rho}{2\epsilon_0}\,s}$$ inside $\quad s<a$ inside

$s>a$, Outside $\quad Q_{enc} = \rho L\int_0^a 2\pi s'\,ds' = \rho\pi a^2 L$

$$\boxed{E(s) = \frac{\rho\, a^2}{2\, s\,\epsilon_0}}$$ outside

3 a) This is a 1 dimensional (z) problem

$V(x, y, z) = V(z)$

Sol'ns of Laplace eqn $V(z) = A + Bz$

$V(z=0) = 0 = A$

$V(z=d) = Bd = V_0$

$B = V_0/d$ so

$V(z) = V_0 z/d$

b) $\vec{E} = -\vec{\nabla} V = -\hat{k} \frac{\partial}{\partial z} V = \boxed{-\frac{V_0}{d} \hat{k} = \vec{E}}$

c) General result $\sigma = \epsilon_0 \vec{E} \cdot \hat{n}$ $\hat{n}$ points out of conductor

σ on lower plate $= -\vec{E} \cdot \hat{k} \epsilon_0$

$= \frac{V_0}{d} \epsilon_0 = \sigma(\text{lower})$

$\sigma(\text{upper}) = -\vec{E} \cdot (-\hat{k}) = \boxed{-\frac{V_0 \epsilon_0}{d} = \sigma(\text{upper})}$

d) Capacitance $C = \frac{Q}{V_0} = \frac{\sigma A}{V_0}$ of positive charge

$$\boxed{C = \frac{\cancel{|V_0|}\epsilon_0 A}{d}}$$

V_0 can be positive or negative but $C > 0$

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Solutions to HW#3

1]. The potential energy of a sphere of charge.

(a) Charge density $\rho = \frac{Q}{\frac{4}{3}\pi R^3}$

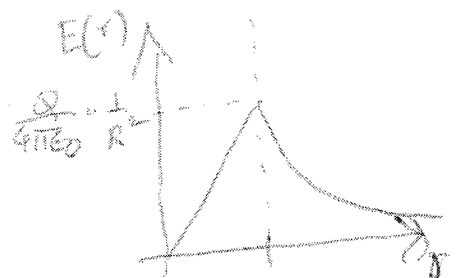
Using symmetry & Gauss' Law:

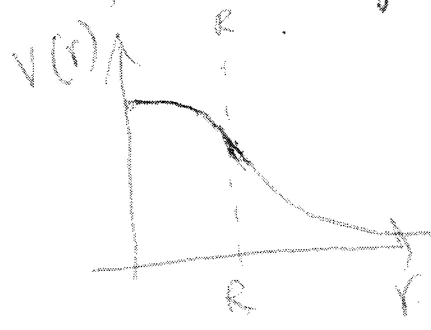
$\vec{E}(\vec{r}) = E(r)\hat{r}$ and

$$E(r) = \begin{cases} \frac{Q}{4\pi\epsilon_0}\frac{1}{r^2} & r > R \\ \frac{Q}{4\pi\epsilon_0}\frac{r}{R^3} & r \le R \end{cases}$$

Electric potential $V(r) = -\int_\infty^r \vec{E}(\vec{r})\cdot d\vec{l}$

$$= -\int_\infty^r E(r)\,dr = \begin{cases} \frac{Q}{4\pi\epsilon_0}\frac{1}{r} & r > R \\ \frac{Q}{8\pi\epsilon_0 R}\left(3 - \frac{r^2}{R^2}\right) & r \le R \end{cases}$$





Electrical potential energy of charged sphere:

$$P.E. = \frac{1}{2}\int_{\text{all space}} \rho V\, d\tau \qquad \rho(r) = \begin{cases} 0 & r > R \\ \rho & r < R \end{cases}$$

$$= \frac{1}{2}\rho\int_0^R V(r)\, 4\pi r^2\, dr$$

$$PE = \frac{1}{2}\rho\cdot\cancel{4\pi}\frac{Q}{\cancel{8\pi}\epsilon_0 R}\int_0^R\left(3 - \frac{r^2}{R^2}\right)r^2\, dr = \frac{1}{2}\frac{Q}{\frac{4}{3}\pi R^3}\cdot\frac{Q}{2\epsilon_0 R}\left[3\cdot\frac{1}{3}R^3 - \frac{1}{5}\frac{R^5}{R^2}\right]$$

$$= \frac{3Q^2}{4\cdot 4\pi\epsilon_0 R^4}\cdot\frac{4}{5}R^3 \Rightarrow \boxed{PE = \frac{3Q^2}{20\pi\epsilon_0 R}}$$

Solns to HW#3

1] (b) Gravitational PE can be calculated in analogy to electric PE. Since the grav. force is attractive, sign is negative & proportionality constant : $\frac{1}{4\pi\epsilon_0} \rightarrow G$; $q_1, q_2 \rightarrow m_1, m_2$.

Thus, $PE_{grav} = -\frac{3M^2}{20\pi\epsilon_0 R'} \times \frac{4\pi\epsilon_0 \times G}{\cancel{\ }}$

$= -\frac{3GM^2}{5R'}$

(c) Moon mass $\approx 7\times10^{22}$ kg ; Moon Radius ≈ 1700 km $\approx 1.7\times10^6$ m

$PE_{grav} = -\frac{3\times 6.7\times10^{-11}\times(7\times10^{22})^2}{5\times1.7\times10^6}$ Joules

$\approx -1.2\times10^{29}$ Joules.

(d). Need $|PE_{elec}| > |PE_{grav}|$ or $PE_{elec} + PE_{grav} > 0$.

then, $\frac{3Q^2}{20\pi\epsilon_0 R} + -\frac{3GM^2}{5R'} > 0$.

Minimum radius needed $= \frac{Q^2}{4\pi\epsilon_0} \cdot \frac{R_{moon}}{GM^2_{moon}}$.

Water density = $1000 kg/m^3$ → take to be proton density

⇒ charge density $\rho = \frac{1000 \ kg/m^3}{m_p} \times e \approx 9.6\times10^{10} \frac{C}{m^3}$

$PE_{elec} = \frac{3Q^2}{20\pi\epsilon_0 R} = \frac{3\rho^2}{20\pi\epsilon_0 R}\left(\frac{4}{3}\pi R^3\right)^2 = \frac{1}{3}\cdot\frac{4}{5}\frac{\rho^2}{\epsilon_0}\pi R^5 = 1.2\times10^{29}$ J

solves this minimum radius $\approx \underline{17 cm}$

Soln to HW#3

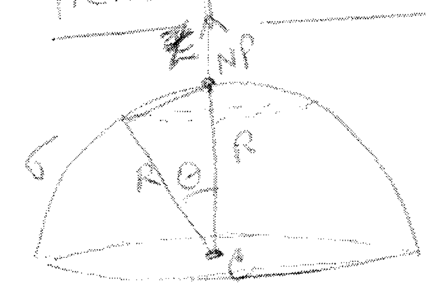
1] (e) Voltage at surface of sphere (see (a)):

$$V(R) = \frac{Q}{4\pi\epsilon_0 R} = \frac{\rho \cdot \frac{4}{3}\pi R^3}{4\pi\epsilon_0 R} = \frac{\rho}{3\epsilon_0} \cdot R^2$$

$$= \frac{9.6\times10^{10}}{3\times 8.85\times10^{-12}} \times (0.17)^2 \text{ Volts}$$

$$V(R) \simeq 10^{20} \text{ Volts.}$$

2] Hemispherical Bowl



Voltage at any point along the vertical z-axis can be obtained by summing up the contributions for rings of charge located at azimuthal coordinate θ (dashed line).

Ring at θ has radius $= R\sin\theta$

Distance from $NP = \sqrt{R^2+R^2-2R^2\cos\theta}$ and from center $= R$.

Potential at north pole (NP):

$$V_{NP} = \frac{1}{4\pi\epsilon_0}\int_0^{\pi/2} \frac{\sigma(2\pi R\sin\theta)\,R\,d\theta}{\sqrt{R^2+R^2-2R^2\cos\theta}}$$

$$= \frac{\sigma R^2}{2\epsilon_0}\cdot\frac{1}{\sqrt{2}R}\cdot\int_0^{\pi/2}\frac{\sin\theta\,d\theta}{\sqrt{1-\cos\theta}}$$

Can integrate by changing variables to $x = \cos\theta$.

$$= \frac{\sigma R}{2\sqrt{2}\,\epsilon_0}\cdot\int_0^1\frac{dx}{\sqrt{1-x}} = \frac{\sigma R}{\sqrt{2}\,\epsilon_0} = V_{NP}.$$