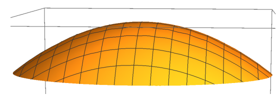
PHYSICS 321- Fall 2019
CLASSICAL ELECTRODYNAMICS

Sep't 26 Problem Set 1 These problems are due 10 am Thursday, Oct 3 in the boxes outside the north entrance of PAB (same as tutorial boxes). Please label your homework with name, section, and HW#.

1. Consider a surface $\mathcal{S}$ bounded at the top by a spherical cap, $x^2 + y^2 + z^2 = 3a^2$, with $z > a$, together with its base: $x^2 + y^2 \leq 2a^2$, $z = a$. See the figure. The vector $\vec{v} = \mathbf{v} = xz\mathbf{i} - yz\mathbf{j} + y^2\mathbf{k} = xz\hat{\mathbf{x}} - yz\hat{\mathbf{y}} + y^2\hat{\mathbf{z}}$. Compute the flux of $\mathbf{v}$ through the surface $\mathcal{S}$ directly (a) and (b) using the divergence theorem (1.56). The divergence in spherical coordinates is given on the inside front cover.



2. The function ϕ is given by $\phi = \frac{1}{r}$, and $\mathbf{v} = -\vec{\nabla}\phi$. Let $\mathcal{P}$ be a curve in the xy plane ($z = 0$) such that $x^2 + y^2 = a^2$. Compute $\oint_{\mathcal{P}} \mathbf{v} \cdot d\mathbf{l}$ directly and using Stokes theorem with the surface $\mathcal{S}$ defined by the hemisphere of radius of radius a, bounded by $\mathcal{P}$, with $z \geq 0$.

3. (a) Evaluate $\int_{\mathcal{V}} \mathbf{r} \cdot (\mathbf{d} - \mathbf{r})\delta(\mathbf{r} - \mathbf{e})$, where $\mathbf{d} = (0, 1, 2)$, $\mathbf{e} = (2, 1, 0)$ and $\mathcal{V}$ is a sphere of radius 2 centered at (1,1,1).

(b) Evaluate the integral $J = \int_{\mathcal{V}} e^{-r/a}(1 + \frac{r}{a})\left(\boldsymbol{\nabla} \cdot \frac{\hat{\mathbf{r}}}{r^2}\right) d\tau$, where $\mathcal{V}$ is a sphere of radius R, centered at the origin. Use two different methods as in Ex. 1.16.

4 (a) Consider a river in which the water velocity v is proportional to the distance from the bottom, and the flow is in the z-direction according to $v_x = 0, v_y = 0, v_z = cx$, where c is a given constant. Find $\boldsymbol{\nabla} \times \mathbf{v}$.

(b) Sketch the field lines (pages 67,68) for the vector function $\mathbf{F}(x, y, z) = \mathbf{i}x + \mathbf{j}y$. Explain why $\mathbf{F}$ does or does not have a divergence and curl.

(c) Sketch the field lines for the vector function$\mathbf{G}(x, y, z) = \frac{-\mathbf{i}y + \mathbf{j}x}{\sqrt{x^2+y^2}}$. Explain why $\mathbf{G}$ does or does not have a divergence and curl.

5. *Maxwell's Equations* (a) Show that the general form of Maxwell's Equations are consistent with the equation $\boldsymbol{\nabla} \cdot \mathbf{J} + \frac{\partial \rho}{\partial t} = 0$.

(b) Use (a) to show that the time rate of change of the charge inside a given fixed volume is equal to the current flowing through the surface. Thus the equation in (a) is a statement of current and charge conservation

Physics 321

HW1 Solutions Prof Miller

(1)  $x^2+y^2+z^2 = 3a^2$

$$\vec{V} = xz\,\hat{i} - yz\,\hat{j} + y^2\hat{k}$$

Flux $= \int_S \vec{V}\cdot\hat{n}\,da$ $\quad$ S = flat part + hemisphere

<u>Flat part</u>: $\quad z = a \quad \hat{n} = -\hat{k}$

$$\vec{V}\cdot\hat{n} = -y^2$$

$$\int_{flat} \vec{V}\cdot\hat{n}\,da = -\int_0^{x^2+y^2<2a^2} y^2\,dx\,dy \quad \text{use symmetry}$$

$$= -\frac{1}{2}\int_0^{\sqrt{2}a}\int_0^{\sqrt{2}a}(x^2+y^2)\,dx\,dy = -\frac{1}{2}\int_0^{\sqrt{2}a} r^2\, r\,dr\,d\phi \text{ (from}$$

polar coordinates) $\quad = -\frac{1}{2}\,2\pi\,4\frac{a^4}{4} = -\pi a^4$

Hemisphere $\quad \hat{n} = \hat{r} = \dfrac{x\hat{i}+y\hat{j}+z\hat{k}}{\sqrt{3}a}$

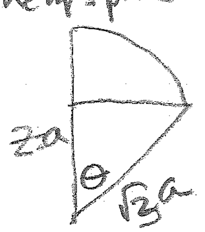
$$\vec{V}\cdot\hat{n} = \frac{1}{\sqrt{3}a}\left[x^2z - y^2z + y^2z\right]$$

$$= \frac{1}{\sqrt{3}a}x^2z = \frac{1}{\sqrt{3}a}(\sqrt{3}a\sin\theta\cos\phi)^2(\sqrt{3}a\cos\theta)$$

$$= 3a^2\sin^2\theta\cos^2\phi\cos\theta$$

$$\int_{hemisphere}\vec{V}\cdot\hat{n}\,da = \int_{\cos\theta=\frac{1}{\sqrt{3}}}^{\cos\theta=1}\sin\theta\,d\theta\int_0^{2\pi}d\phi\,(\sqrt{3}a)^2\left[3a^2\sin^2\theta\cos^2\phi\cos\theta\right]$$

hemisphere sideview



in spherical coordinates

$da = r^2\sin\theta\,d\theta\,d\phi$

let $x = \cos\theta$

$$\int_{\text{hemisphere}} \vec{v}\cdot\hat{n}\,da = 3a^2 \cdot 3a^2 \int_{\frac{1}{\sqrt{3}}}^{1} dx\,(1-x^2)x \int_0^{2\pi} d\phi \cos^2\phi$$

$$= 9a^4 \left[\frac{x^2}{2} - \frac{x^4}{4}\right]_{\frac{1}{\sqrt{3}}}^{1} \cdot \frac{1}{2}\, 2\pi$$

$$= 9a^4 \left[\frac{1}{2} - \frac{1}{4} - \frac{1}{6} + \frac{1}{4}\cdot\frac{1}{9}\right]\pi = \pi a^4$$

Total flux $= 0$ $\quad (-\pi a^4 + \pi a^4) = 0$

Now use divergence theorem

$$\vec{\nabla}\cdot\vec{V} = \frac{\partial}{\partial x}(xz) + \frac{\partial}{\partial y}(yz) + \frac{\partial}{\partial z} y^2$$

$$= z - z = 0 \quad !$$

$$\int \vec{\nabla}\cdot\vec{v}\, d\tau = 0$$

So Divergence thm is satisfied

Much easier to use $\vec{\nabla}\cdot\vec{v}$ directly

(2.) $\phi = \frac{1}{\sqrt{x^2+y^2+z^2}} = \frac{1}{r}$ In spherical coordinates

$\vec{v} = -\vec{\nabla}\phi = \frac{\hat{r}}{r^2}$ (note that we will not be using region with $r=0$)

P is path on circle

$$\oint_P \vec{v}\cdot d\vec{l} = \oint_P (-)\vec{\nabla}\phi \cdot d\vec{l} = -\oint_P d\phi$$

$= 0$ (beginning and end of closed path is at same point)

How about Stokes theorem?

$\vec{\nabla}\times\vec{v} = 0$ because $\vec{\nabla}\times\vec{\nabla} = 0$ when acting on any function. Thus

$$\int \vec{\nabla}\times\vec{v}\cdot d\vec{a} = 0$$

3. a) First determine if $\vec{e}$ is inside the volume $\mathcal{V}$

The center of the sphere is at

$\vec{p} = (1,1,1)$

$\vec{e} - \vec{p} = (1, 0, -1)$, length $|\vec{e} - \vec{p}| = \sqrt{1+1} = \sqrt{2}$

$\sqrt{2} < 2$ (radius of shere)

$\vec{e}$ is inside $\mathcal{V}$ so

$$I = \int_{\mathcal{V}} \vec{r} \cdot (\vec{d} - \vec{r})\, \delta(\vec{r} - \vec{e}) = \vec{e} \cdot (\vec{d} - \vec{e})$$

$$= \vec{e} \cdot \vec{d} - e^2$$

$$\vec{e} \cdot \vec{d} = [2\ 1\ 0] \cdot \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix} = 1, \quad e^2 = 2$$

So $I = 1 - 2 = -1$

b) i) $\vec{\nabla} \cdot \frac{\hat{r}}{r^2} = -\nabla^2 \frac{1}{r} = +4\pi\delta(\vec{r})$ then

$$J = \int_{\mathcal{V}} e^{-r/a} \left(1 + \frac{r}{a}\right) \left(-4\pi \delta(\vec{r})\right) = +4\pi$$

ii) Use 1.59 $\int f \vec{\nabla} \cdot \vec{A}\, d\tau = -\int \vec{A} \cdot \vec{\nabla} f\, d\tau + \oint_S f \vec{A} \cdot d\vec{a}$

here $f = e^{-r/a}(1 + r/a)$

$\vec{A} = \frac{\hat{r}}{r^2}$

so $J = -\int \frac{\hat{r}}{r^2} \cdot \vec{\nabla} f\, d\tau + \int_S f \frac{1}{r^2}\, da$

$$\vec{\nabla} f = \hat{r}\frac{\partial}{\partial r} f = \left(-\frac{1}{a}\left(1+\frac{r}{a}\right) + \frac{1}{a}\right)e^{-r/a}$$

$$= -r/a^2 \, e^{-r/a}$$

$$J = \int_0^R \frac{r}{a^2} e^{-r/a} \, 4\pi \frac{r^2 \, dr}{r^2} + \oint_S \left(1+\frac{r}{a}\right)\frac{e^{-r/a}}{r^2} \, da$$

$$= \frac{4\pi}{a^2}\int_0^R r \, e^{-r/a} \, dr + 4\pi\left(1+\frac{R}{a}\right)\frac{e^{-R/a}}{\cancel{R^2}} \cancel{R^2}$$

$$= \frac{4\pi}{a^2}\left[a\left(a - e^{-R/a}(a+R)\right)\right] + 4\pi\left(\frac{a+R}{a}\right)e^{-R/a}$$

$= 4\pi$ as expected

4 a) $\vec{v} = cx\,\hat{k}$

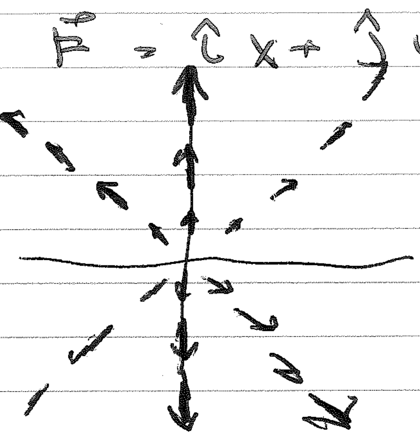
$\vec{\nabla}\times\vec{v}$ in y direction

$$v_2 = \epsilon_{213}\frac{\partial v_3}{\partial x_1} + \epsilon_{231}\frac{\partial v_1}{\partial x_3}$$

$$= (-1)\frac{\partial v_z}{\partial x} + (1)\frac{\partial v_x}{\partial z}$$

$$= -c \quad ; \quad \vec{\nabla}\times\vec{v} = -c\hat{y}$$

b) $\vec{F} = \hat{i}\,x + \hat{j}\,y$



Field lines: direction gives direction of vector. Mag. is mag of vector and density of lines

$$\vec{\nabla}\cdot\vec{F} = \frac{\partial F_x}{\partial x} + \frac{\partial F_y}{\partial y} = 2$$

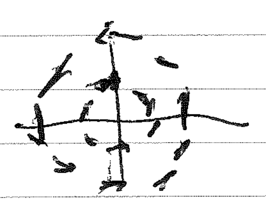
$\vec{\nabla}\times\vec{F} = 0$ no swirling or

$$\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x & y & 0 \end{vmatrix} = \hat{i}\left(\frac{\partial 0}{\partial y} - \frac{\partial z}{\partial y}\right) - \hat{j}\left(\frac{\partial}{\partial z}0 - \frac{\partial}{\partial z}y\right)$$

$$+ \hat{k}\left(\frac{\partial y}{\partial x} - \frac{\partial x}{\partial y}\right) = 0$$

c) $\vec{G} = \dfrac{-\hat{i}y + \hat{j}x}{\sqrt{x^2+y^2}}$ $\quad |\vec{G}| = 1$

$r = \sqrt{x^2+y^2}$



$$\vec{\nabla}\times\vec{G} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ -\frac{y}{r} & \frac{x}{r} & 0 \end{vmatrix} = \hat{k}\left(\frac{\partial}{\partial x}\frac{x}{r} + \frac{\partial}{\partial y}\frac{y}{r}\right)$$

$$\frac{\partial r}{\partial x} = \frac{x}{r} \qquad \frac{\partial r}{\partial y} = \frac{y}{r}$$

$$\vec{\nabla} \times \vec{G} = \left[\frac{1}{r} - \frac{x^2}{r^3} + \frac{1}{r} - \frac{y^2}{r^3} \right] \hat{k} = \frac{2}{r} - \frac{r^2}{r^3} \hat{k} = \frac{1}{r} \hat{k}$$

$$\vec{\nabla} \cdot G = -\frac{\partial}{\partial x} \frac{y}{r} + \frac{\partial}{\partial y} \frac{x}{r}$$

$$= -y \left(-\frac{x}{r^3} \right) + x \left(-\frac{y}{r^3} \right) = 0$$

5 $\vec{\nabla} \cdot \vec{E} = \rho / \epsilon_0$ Maxwell's eq

$$\vec{\nabla} \cdot \vec{B} = 0$$

$$\vec{\nabla} \times \vec{B} = -\frac{\partial \vec{E}}{\partial t}$$

$$\vec{\nabla} \times \vec{E} = \mu_0 \vec{J} + \mu_0 \epsilon_0 \frac{\partial \vec{B}}{\partial t}$$

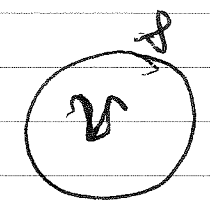
5 a) Use $\vec{\nabla}\times\vec{B} = \mu_0\vec{J} + \mu_0\epsilon_0\frac{\partial \vec{E}}{\partial t}$

Take $\vec{\nabla}\cdot$ of above use $\vec{\nabla}\cdot\vec{E} = \frac{\rho}{\epsilon_0}$

$$\vec{\nabla}\cdot(\vec{\nabla}\times\vec{B}) = \mu_0\left(\vec{\nabla}\cdot\vec{J} + \epsilon_0\frac{\partial}{\partial t}\vec{\nabla}\cdot\vec{E}\right)$$

$$\vec{\nabla}\cdot(\vec{\nabla}\times \text{any vector} = 0) = \mu_0\left(\vec{\nabla}\cdot\vec{J} + \frac{\partial \rho}{\partial t}\right)$$

$$\because \frac{\partial}{\partial x_i}\epsilon_{ijk}\frac{\partial}{\partial x_j}V_j = 0$$

b) 

Take fixed volume $\mathcal{V}$

Take

$$\underbrace{\int_{\mathcal{V}} \frac{\partial \rho}{\partial t}\, d\tau}_{\text{LHS}} = \underbrace{-\int_{\mathcal{V}} \vec{\nabla}\cdot\vec{J}\, d\tau}_{=\text{RHS}}$$

$$\text{LHS} = \frac{d}{dt}\int_{\mathcal{V}} \rho\, d\tau = -\frac{dQ}{dt}$$

RHS use divergence theorem

$$-\int_{\mathcal{V}} \vec{\nabla}\cdot\vec{J}\, d\tau = -\oint_{S} \vec{J}\cdot d\vec{a} = \text{current flowing out}$$