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1. A thin rod of length L has its left end at $x = -L/2$ and its right end at $x = L/2$. The rod carries a line charge density given by $\lambda(x) = \lambda_0 \frac{x^3}{L^3}$.

(a) (5) Consider a specific point $P = \mathbf{r} = (x, y, z) = (L, 2L, 3L)$. Determine the distance between the point P and the point where the line charge density is maximum.

$\lambda(x)$ is max at $x = \frac{L}{2}$, $y=0$, $z=0$

The distance is $\left(\left(L-\frac{L}{2}\right)^2 + 4L^2 + 9L^2\right)^{1/2}$

$= L\left(\frac{1}{4}+4+9\right)^{1/2} = L\left(13\frac{1}{4}\right)^{1/2}$

(b) (10) Determine the electric potential $V(\mathbf{r})$ at all points in space. You can express your answer in terms of a well-defined, one-dimensional integral.

$$V(\vec{r}) = \frac{1}{4\pi\epsilon_0}\int_{-L/2}^{L/2} \frac{\lambda(x')\,dx'}{|\vec{r}-\vec{x}'|}$$

$|\vec{r}-\vec{r}'| = \sqrt{r^2+r'^2-2\vec{r}\cdot\vec{r}'}$

$= \sqrt{r^2+r'^2-2xx'}$

$r'^2 = x'^2$

x is the x component of the $\vec{r}$ vector

$$V(\vec{r}) = \frac{1}{4\pi\epsilon_0}\frac{\lambda_0}{L^3}\int_{-L/2}^{L/2}\frac{(x')^3\,dx'}{\sqrt{r^2+x'^2-2xx'}}$$

(c) (10) Determine the electric field at the origin.

$$\vec{E}(\vec{r}) = \frac{1}{4\pi\epsilon_0}\frac{\lambda_0}{L^3}\int_{-L/2}^{L/2}\frac{(\vec{r}-x'\hat{\imath})\,dx'\,x'^3}{\left[r^2+x'^2-2xx'\right]^{3/2}}$$

$$\vec{E}(0) = \frac{\lambda_0\hat{\imath}}{4\pi\epsilon_0 L^3}\int_{-L/2}^{L/2}\frac{(-)x'^4\,dx'}{|x'|^3}$$

$$\vec{E}(0) = \frac{\lambda_0}{4\pi\epsilon_0}\frac{\hat{\imath}}{L^3}(-)2\int_0^{L/2}x'\,dx' = \frac{\lambda_0\hat{\imath}(-2)}{4\pi\epsilon_0 L^3}\,\frac{1}{2}\,\frac{L^2}{4} = \frac{-\lambda_0\hat{\imath}}{16\pi\epsilon_0}\frac{1}{L}$$

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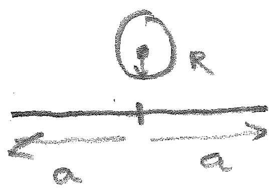
2. This problem is concerned with a thin disk of radius a with a total surface charge $Q > 0$. The surface charge density is positive for all points on the disk.

(a) (10) Consider an imaginary surface (a Gaussian sphere) of radius $R < a$ which has a center on the axis of the disk, at a height $z > R$ above the center of the disk. Determine the electric flux through the Gaussian sphere.



The sphere is above the disk. No charge is enclosed. The electric flux thru the sphere is $\boxed{0.}$

(b) (5) Suppose the surface charge density takes on the form $\sigma(s) = \alpha s$, where s is the distance from the center, and $\alpha > 0$. Determine the constant α in terms of the given constants.

$$Q = \int da\, \sigma = \int_0^a s\,ds(\alpha s) \int_0^{2\pi} d\phi$$

$$= \frac{2\pi \alpha a^3}{3} \; ; \quad \boxed{\alpha = \frac{3Q}{2\pi a^3}}$$

(c) (10) Again consider the $\sigma(s)$ of part (b), the electric potential $V(z)$ at a distance z on the axis of the disk (directly above the center of the disk) is given by

$$V(z) = \frac{\alpha}{2\epsilon_0} \int_0^a dr' \frac{r'^2}{\sqrt{z^2 + r'^2}}.$$

Determine the electric field $\vec{E}(z)$ in terms of a well-defined one-dimensional integral.

From cylindrical symmetry the horizontal component of $\vec{E}$ vanish. Thus $\vec{E} = E(z)\,\hat{z}$ with $E(z) = -\frac{\partial V}{\partial z} = \frac{\alpha}{4\epsilon_0} \int_0^a dr' \frac{r'^2\, z}{[z^2 + r'^2]^{3/2}}$

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3. A region of space contains a spherically symmetric charge density $\rho(r)$, where r is the distance from a specified origin.

(a) (5) Consider an imaginary sphere (Gaussian sphere) of radius R, centered at the origin. Determine the amount of charge (define it as $Q(R)$) within the Gaussian sphere. Your answer should be given in terms of a well-defined mathematical operation which involves the given $\rho(r)$.

$$Q(R) = 4\pi \int_0^R r^2 dr \, \rho(R)$$

(b) (8) Now determine the electric field $\vec{E}$ at positions on the Gaussian sphere. Express your answer in terms of $Q(R)$, which you may now assume as given. Be sure to specify and explain the direction of $\vec{E}$. $\vec{E}$ points in the $\hat{r}$ direction because of spherical symmetry, Gauss Law: $\oint \vec{E} \cdot d\vec{a} = \frac{q(\text{enclosed})}{\epsilon_0}$

so $4\pi R^2 E(R) = \frac{1}{\epsilon_0} Q(R)$

$$E(R) = \frac{Q(R)}{4\pi\epsilon_0 R^2}$$

(c) (7) Now we take $\rho(r) = \frac{3q_0}{4\pi R_0^3}$ if $r < R_0$ and 0 for $r \geq R_0$ Here R_0 is a given length and q_0 a given charge. Determine the electric field $\vec{E}$ at a position r away from the origin.

If $r \geq R_0$ $\quad \vec{E}(\vec{r}) = \hat{r}\frac{q_0}{4\pi r^2}$

if $r < R_0$, $\vec{E}(\vec{r}) = \hat{r}\frac{1}{4\pi\epsilon_0 r^2} 4\pi \int_0^r r'^2 dr' \frac{3 q_0}{4\pi R_0^3} = \frac{\vec{r}\, q_0}{4\pi\epsilon_0 R_0^3}$

(d) (5) Consider position P defined, by the coordinates $(x, y, z) = (2a, a/2, 0)$. Given a general spherically symmetric charge distribution $\rho(r)$, determine $\nabla^2 V(\mathbf{r})$ at P.

$\nabla^2 V = -\rho(r)/\epsilon_0$ $\qquad$ For P, $r = \sqrt{4a^2 + a^2/4}$ $= \sqrt{5/4}\, a$

$$\boxed{\nabla^2 V = -\frac{\rho\left(\sqrt{\frac{5}{4}}a\right)}{\epsilon_0}}$$