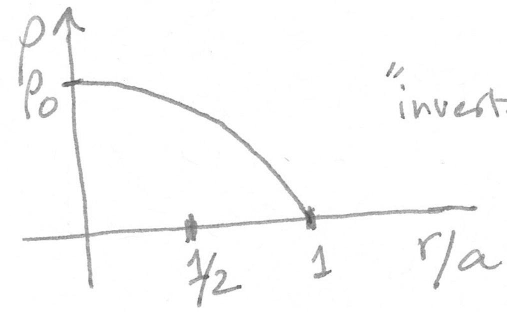
Solns to HW#2

4]. Field of an Atomic Nucleus



"inverted parabola".

$a = 3.4\,fm$.

(a). Nucleus has 21 protons, what is ρ_0?

Total charge in nucleus $= 21e$ where $e = 1.6\times10^{-19}\,C$.

$$= \int_{\text{volume}} \rho\, dV = \rho_0 \int_0^a (1 - r^2/a^2)\, 4\pi r^2\, dr + \int_a^\infty 0\, dV$$

$$= \rho_0 4\pi \left(\int_0^a r^2 dr - \frac{1}{a^2}\int_0^a r^4\, dr \right) = \rho_0 \cdot 4\pi \left(\frac{1}{3}a^3 - \frac{1}{5}\frac{a^5}{a^2} \right).$$

$$= 4\pi \rho_0 \cdot \frac{2}{15} a^3 = 21e.$$

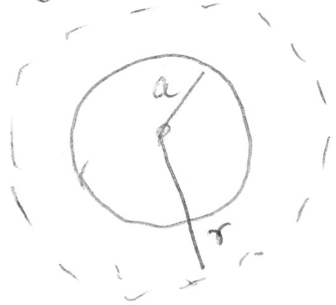
$$\Rightarrow \rho_0 = \frac{21e.}{4\pi \times \frac{2}{15} a^3} = \frac{21\times15}{4\pi\times2} \times \frac{1.6\times10^{-19}}{(3.4\times10^{-15})^3}\ \frac{C}{m^3}.$$

$$\boxed{\rho_0 \simeq 5.1\times10^{25}\ \text{Coulombs}/(\text{meter})^3}$$

(b) The spherical symmetry allows us to use Gauss' Law effectively

For sphere radius $r > a$ and centered at the center of the nucleus, we apply Gauss' Law:

4] (b) cont'd.



By symmetry: $\vec{E}(\vec{r}) = E(r)\hat{r}$.

$$\oint_{\text{surface of sphere}} \vec{E}\cdot d\vec{A} = \frac{Q_{enc}}{\epsilon_0}$$ [Gauss' Law Integral Form].

$$E(r)\,4\pi r^2 = \frac{1}{\epsilon_0}\int_0^a \left[\rho(r')4\pi r'^2\right]dr'$$

$\rho = 0$ ~~all~~ outside nucleus.

$$= \frac{1}{\epsilon_0}\cdot 21e$$

$$\Rightarrow \boxed{E(r) = \frac{1}{4\pi\epsilon_0}\cdot\frac{21e}{r^2}}$$

$r > a$. $\Rightarrow \vec{E}(\vec{r}) = E(r)\hat{r}$.

$$V(r) = -\int_\infty^r E(r')\,dr' = -\frac{1}{4\pi\epsilon_0}\cdot 21e\int_\infty^r \frac{dr'}{r'^2} = \frac{-21e}{4\pi\epsilon_0}\left(\frac{-1}{r'}\right)\Bigg|_\infty^r$$

$$\boxed{V(r) = \frac{21e}{4\pi\epsilon_0}\cdot\frac{1}{r}}$$ for $r > a$.

At surface $r = a$,
$$\boxed{\begin{aligned}\vec{E}(a) &= \frac{1}{4\pi\epsilon_0}\cdot\frac{21e}{a^2}\hat{r}\\ V(a) &= \frac{21e}{4\pi\epsilon_0 a}\end{aligned}}$$

(c) Inside nucleus, use same approach as in (b) but with sphere radius $r < a$

$$E(r)\,4\pi r^2 = \frac{1}{\epsilon_0}\int_0^{\cancel{a}\, r} \rho(r')4\pi r'^2\,dr'$$

4] (c) Cont'd.

$$E(r)\,4\pi r^2 = \frac{1}{\epsilon_0}\cdot \rho_0 4\pi\left[\frac{1}{3}r^3 - \frac{1}{5}\frac{r^5}{a^2}\right].$$

$$= \frac{\rho_0 4\pi}{\epsilon_0}\cdot r^3\left[\frac{1}{3} - \frac{1}{5}\left(\frac{r}{a}\right)^2\right]$$

$$\Rightarrow \boxed{E(r) = \frac{\rho_0 r}{\epsilon_0}\left[\frac{1}{3} - \frac{1}{5}\left(\frac{r}{a}\right)^2\right]}$$ for $r<a$ & $\vec{E}(\vec{r}) = E(r)\hat{r}$

and $$V(r) = -\int_\infty^r E(r')\,dr'$$

$$= -\int_\infty^a E(r')\,dr' - \int_a^r E(r')\,dr'$$

$$= \frac{21e}{4\pi\epsilon_0 a} - \frac{\rho_0}{\epsilon_0}\left[\frac{1}{3}\int_a^r r'\,dr' - \frac{1}{5a^2}\int_a^r r'^3\,dr'\right]$$

$$\boxed{V(r) = \frac{21e}{4\pi\epsilon_0 a} - \frac{\rho_0}{\epsilon_0}\left[\frac{1}{6}(r^2-a^2) - \frac{1}{20a^2}(r^4-a^4)\right]}$$ for $r<a$

(d) At center of nucleus, $r=0$

then $\boxed{E(0) = 0}$

$$V(0) = \frac{21e}{4\pi\epsilon_0 a} - \frac{\rho_0}{\epsilon_0}\left[-\frac{a^2}{6} + \frac{a^4}{20a^2}\right]$$

from (a): $21e = 4\pi\rho_0\frac{2}{15}a^3$

$$= \frac{\rho_0}{\epsilon_0}\frac{2}{15}a^2 + \frac{\rho_0 a^2}{\epsilon_0}\left[\frac{1}{6} - \frac{1}{20}\right].$$

$$\boxed{V(0) = \frac{\rho_0}{\epsilon_0}a^2\cdot\frac{1}{4}}$$

4] (d). Maximum of $E(r)$ where (here $\vec{E}(\vec{r}) = E(r)\hat{r}$).

$$E(r) = \begin{cases} \dfrac{1}{4\pi\epsilon_0} \cdot \dfrac{2Ie}{r^2} = \dfrac{\rho_0}{\epsilon_0}\dfrac{2}{15}\dfrac{a^3}{r^2} & (r > a) \\[2ex] \dfrac{\rho_0 r}{\epsilon_0}\left[\dfrac{1}{3} - \dfrac{1}{5}\left(\dfrac{r}{a}\right)^2\right] & (0 < r < a) \end{cases}$$

fields drops as $\frac{1}{r^2}$ for $r > a$, so max. must be in the region $0 < r \leq a$.

Take usual approach & set $\dfrac{\partial E}{\partial r} = 0$ to get:

$$\left[\frac{1}{3} - \frac{1}{5}\frac{r^2}{a^2}\right] + r\left[-\frac{2}{5}\frac{r}{a^2}\right] = 0$$

$$\Rightarrow \frac{3}{5}\left(\frac{r}{a}\right)^2 = \frac{1}{3} \Rightarrow \boxed{\frac{r}{a} = \sqrt{\frac{5}{9}} = \frac{\sqrt{5}}{3}} \approx 0.75$$

By inspection, $\left.\dfrac{\partial^2 E}{\partial r^2}\right|_{r = \frac{\sqrt{5}}{3}a} < 0$, therefore $\dfrac{r}{a} = \dfrac{\sqrt{5}}{3}$ is the radial location of maximum electric field magnitude.

(e). see expression in 4(d) & earlier.