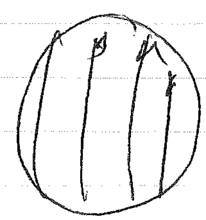
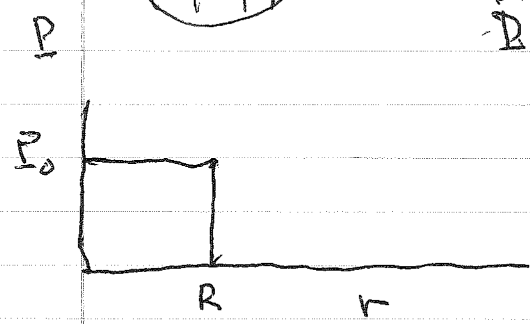
Example showing $\vec{\nabla} \cdot \vec{P}$ get surface term



$\vec{P}$ = constant inside sphere of radius R, 0 outside

$$\vec{P} = P_0 \hat{k}\, \theta(R-r)$$



$$\vec{\nabla} \cdot \vec{P} = \frac{\partial}{\partial z} P_0\, \theta(R-r)$$

$$= \frac{\partial r}{\partial z} \frac{d}{dr} P_0\, \theta(R-r)$$

$$= \frac{z}{r} P_0 \frac{d}{dr} \theta(R-r)$$

$$= \cos\theta\,(-)\,\delta(R-r)$$

Divergence of $\vec{P}$ includes surface delta terms

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$$\vec{\nabla}\cdot\vec{D} = \rho_f$$

3 electric field vector $\vec{E}$ $\vec{P}$ D

integrate both side so over vol

$$\int_V d^3r\, \vec{\nabla}\cdot\vec{D} = \int \rho_f\, d^3r$$

$$\int_S d\vec{a}\cdot\vec{D} = Q_f$$ free charge enclosed

Very powerful. free charges stuff we can control

given $\rho_f \rightarrow \rho_b$

Note there is no ϵ_0

Special but typical case

Linear dielectric

$\vec{P} \propto \vec{E}$ makes sense from examples we've seen

$\vec{P} = \epsilon_0 \chi_e \vec{E}$ $\chi_e > 0$

$\chi_e =$ electric susceptibility (no dimensions)

Then $\vec{D} = \epsilon_0\vec{E} + \vec{P} = \epsilon_0[1+\chi_e]\,\vec{E}$

$\vec{D} = \epsilon\vec{E}$ with $\epsilon = \epsilon_0(1+\chi_e)$

ϵ is called the permittivity

ϵ_0 is permittivity of free space $\chi_e > 0$

$\frac{\epsilon}{\epsilon_0} = 1+\chi_e \equiv \epsilon_r$ relative permittivity or dielectric constant

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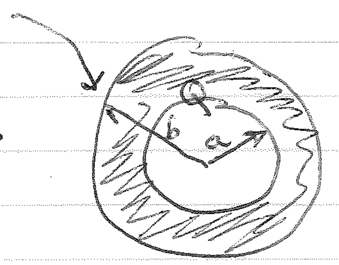
The "different kinds" of charges bound & free and the different fields $\vec{D}$ & $\vec{E}$ are kind of confusing at first

Nice Example from book

b outer radius



Metal sphere radius a carries charge Q surrounded by linear dielectric of permittivity ϵ. Find potential at center relative to infinity

To compute V need to know $\vec{E}$

to find $\vec{E}$ we need the bound charge, so need to know $\vec{P}$, to know $\vec{P}$ need $\vec{E}$? What do we know?

Know free charge therefore know $\vec{D}$

$\vec{D} = \dfrac{Q}{4\pi r^2}\hat{r}$ by Gauss's Law

$\vec{D}$ only cares about free charge

Given $\vec{D}$ & linear medium can get $\vec{E}$

if $r > b$ $\quad \vec{D} = \epsilon_0 \vec{E} \quad \vec{E} = \dfrac{Q\hat{r}}{4\pi\epsilon_0 r^2}$

if $b > r > a$ $\quad \vec{D} = \epsilon \vec{E}$, $\vec{E} = \dfrac{Q\hat{r}}{4\pi\epsilon r^2}$

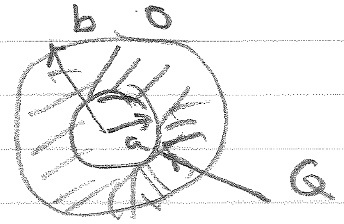
$\vec{E} = 0$ if $r < a$

$$V_{(0)} - V(\infty) = -\int_{\infty}^{0} d\vec{r} \cdot \vec{E} = +\int_{0}^{\infty} d\vec{r} \cdot \vec{E} = \left(\int_{a}^{b} + \int_{b}^{\infty}\right) \vec{E} \cdot d\vec{r}$$

$$= \int_{a}^{b} + \int_{b}^{\infty} dr\, E(r) = \int_{a}^{b} dr \frac{Q}{4\pi \epsilon r^2} + \int_{b}^{\infty} dr \frac{Q}{4\pi \epsilon_0 r^2}$$

$$= \frac{Q}{4\pi} \left[\frac{1}{\epsilon} \left(\frac{1}{a} - \frac{1}{b} \right) + \frac{1}{\epsilon_0} \frac{1}{b} \right]$$

Discuss BC



$$\vec{D} = \frac{Q}{4\pi r^2}\hat{r}$$ everywhere

$r \geq b$ $\quad \vec{E} = \frac{\vec{D}}{\epsilon_0}$ $\quad$ b surface spheres

$b \geq r \geq a$ $\quad \vec{E} = \frac{\vec{D}}{\epsilon}$ $\quad$ tangential component of $\vec{E} = 0$

$$\vec{P} = \epsilon_0 \chi_e \vec{E}$$

$$\vec{P}(r=a) = (\epsilon - \epsilon_0)\vec{E}$$

$\sigma_b = \vec{P}\cdot\hat{n}$, $\hat{n}$ points <u>out</u> of surface

at $r = a$, $\hat{n} = -\hat{r}$

at $r = b$ $\quad \hat{n} = \hat{r}$

$$\sigma_b(r=a) = -(\epsilon - \epsilon_0)\frac{Q}{4\pi\epsilon a^2}$$

$$\sigma_b(r=b) = +(\epsilon - \epsilon_0)\frac{Q}{4\pi\epsilon b^2}$$

Same $|Q_b|$

$\vec{\nabla}\cdot\vec{D} = 0$

$r > a$

on inner surface

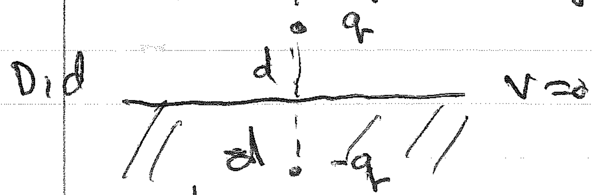
$$\sigma_t = \sigma_b(r=a) + \sigma_f(r=a)$$

$$= +\frac{(\epsilon_0 - \epsilon)}{\epsilon}\frac{Q}{4\pi a^2} + \frac{Q}{4\pi a^2} = \frac{\epsilon_0}{\epsilon}\frac{Q}{4\pi a^2}$$

polarization reduces total charge - screening

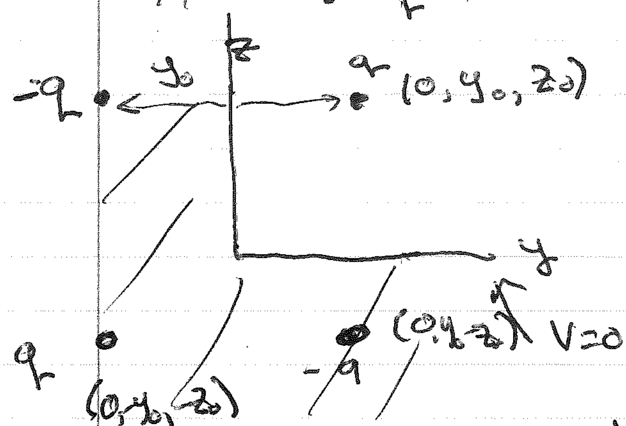
z

Image charge problems



$\nabla^2 V = -\rho/\epsilon_0$, $\rho = q\delta(z-d)\delta(x)\delta(y)$

$V=0$ on plane



need 3 image charges

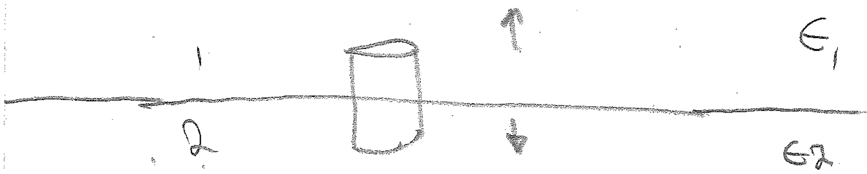
Total image charge must be $-q$

How does V vary at large distances?

Boundary Conditions $\vec{D} = \epsilon_0\vec{E} + \vec{P}$

on $\vec{D}$ and $\vec{E}$ $\quad \vec{\nabla} \cdot \vec{D} = \rho_f$

$$\oint \vec{D} \cdot \hat{n}\, da = Q_f \qquad \vec{\nabla} \cdot \vec{E} = \rho/\epsilon_0$$

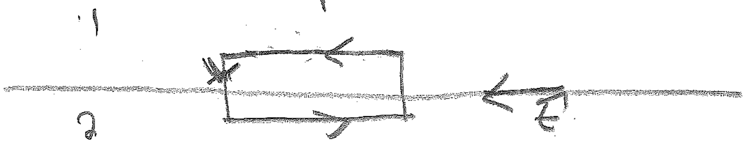


Surface

$$(\vec{D}_1 - \vec{D}_2) \cdot \hat{n} = \sigma_f \qquad (1)$$

what can we say about $\vec{E}$

$$\vec{\nabla} \times \vec{E} = 0$$



$$\oint \vec{E} \cdot d\vec{l} = 0$$

$$\vec{E}_1 \cdot \vec{t} - \vec{E}_2 \cdot \vec{t} = 0 \qquad (2)$$

& tangential component of $\vec{E}$ are continuous

What about V potential? $\vec{E} = \vec{\nabla} V$

must be continuous $V_1 = V_2$ allways

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Need Condition on $\vec{E}$ $(-\nabla V)$. Use $\Box$ previous page

Outside = region 1   Inside = region 2

$$\sigma_f = 0$$

$$-\epsilon_0 \frac{\partial V_{out}}{\partial r} + \epsilon \frac{\partial V_{in}}{\partial r}\bigg|_{r=R} = 0$$

$$-\epsilon_0\left(-E_0 - \frac{2B_1}{R^3}\right) + \epsilon A = 0$$ ∴ 2 eqns 2 unknowns

$$-\epsilon_0\left(-E_0 - \frac{2B_1}{R^3}\right) + \epsilon\left(-E_0 + \frac{B_1}{R^3}\right) = 0$$

$$(\epsilon_0 - \epsilon)E_0 = \frac{B_1}{R^3}(-2\epsilon_0 - \epsilon)$$

$$B_1 = \frac{(\epsilon - \epsilon_0)}{2\epsilon_0 + \epsilon} E_0 R^3$$

$$A = -E_0 + B_1/R^3$$

$$= -E_0 + \frac{(\epsilon - \epsilon_0)}{2\epsilon_0 + \epsilon} E_0$$

$$= E_0\left[\frac{-2\epsilon_0 - \cancel{\epsilon} + \cancel{\epsilon} - \epsilon_0}{2\epsilon_0 + \epsilon}\right]$$

$$= -\frac{3\epsilon_0}{2\epsilon_0 + \epsilon} E_0$$ $\vec{E}$ is reduced in sphere

What limits to check

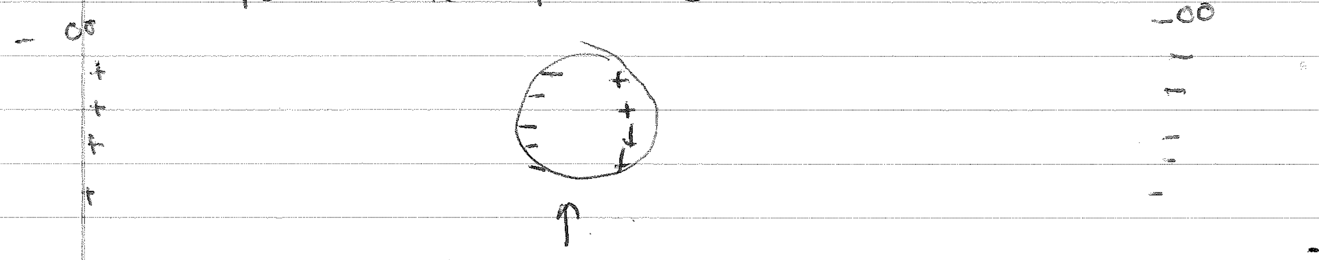
$\epsilon = \epsilon_0$ $\quad B_1 = 0 \quad A_1 = -E_0$ as expected

no dielectrics

what about $\epsilon \to \infty$

$A \to 0$ $\quad \vec{E}$ inside is 0

dipoles line up to cancel $\vec{E}$



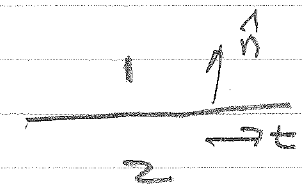
in general reduction of $\vec{E}$

$\epsilon \to \infty$ no $\vec{E}$ inside

like conductor

Linear Dielectric

$$\vec{D} = \epsilon \vec{E}$$
$$= -\epsilon \vec{\nabla} V$$

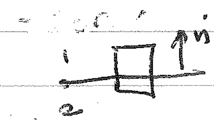


The B.C. on D becomes

$$\hat{n} \cdot \left(-\epsilon_1 \vec{\nabla} V_1 + \epsilon_2 \vec{\nabla} V_2\right) = \sigma_f$$

or $$\boxed{-\epsilon_1 \frac{\partial V_1}{\partial n} + \epsilon_2 \frac{\partial V_2}{\partial n} = \sigma_f}$$

but



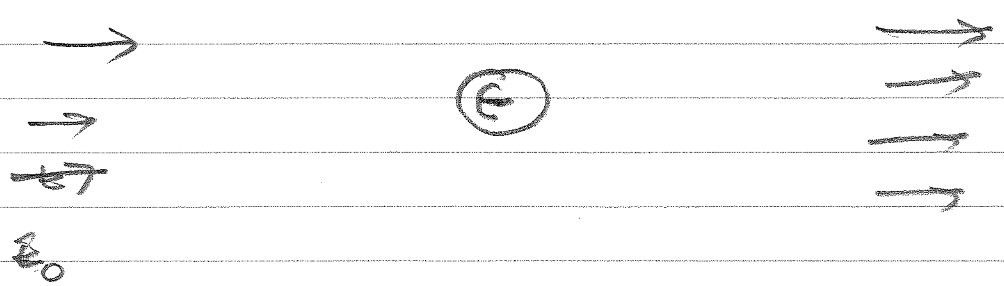
$$\vec{E}_1 \cdot \hat{n} - \vec{E}_2 \cdot \hat{n} = \frac{\sigma}{\epsilon_0}$$

V is continuous across the boundary but $\vec{E}$ is not

Example

$\vec{E} = E_0 \hat{z}$, $V = -E_0 z = -E_0 r \cos\theta$

Now put dielectric sphere radius R

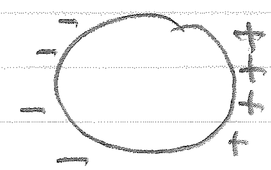


$\epsilon > \epsilon_0$

what happens - get polarization

bound charge density is created

Sphere looks like



dipole moment is created

Want $\vec{E}$ outside and inside

outside $V = -E_0 r\cos\theta + \sum_{l=0} \frac{B_l}{r^{l+1}} P_l(\cos\theta)$

But $\cos\theta = P_1(\cos\theta)$

The $P_{l\neq 1}$ are orthogonal to P_1

out $V_{out} = -E_0\, r\cos\theta + \frac{B_1}{r^2}\cos\theta$

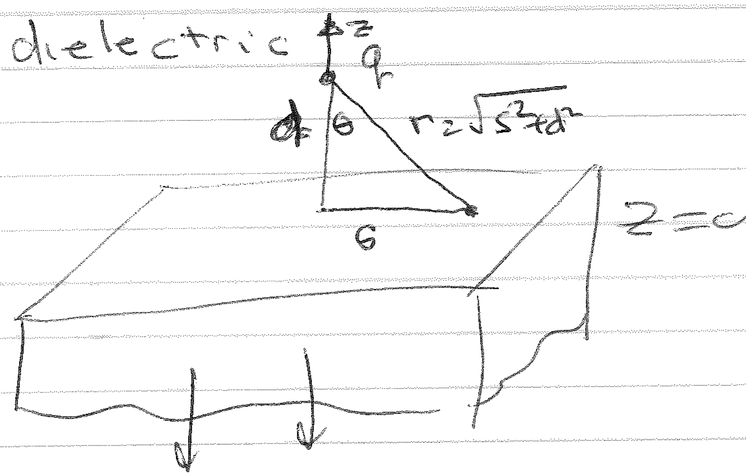
inside $V_{in} = A\, r\cos\theta$

V is continuous at $r = R$

$$-E_0 R + \frac{B_1}{R^2} = A R$$

Need another equation

Boundary value problem with dielectric



region $z<0$ filled with linear dielectric χ_e. Put point charge q a distance d above plane.

? What is force on q. Force of σ_b on q

we can get direction : expect dipoles to be formed

$\circ q$

$\overline{\quad - \quad - \quad - \quad - \quad - \quad}$

→ Attractive force. There is a bound surface charge

$$\sigma_b = \vec{P}\cdot\hat{n} \quad ; \quad \vec{P} = \epsilon_0 \chi_e \vec{E}$$

$\hat{n} = $ up

$$\sigma_b = \epsilon_0 \chi_e E_z$$

What is E_z? part comes from point charge I and part from σ_b itself. II

(I) from point charge $E_z^{(q)} = -\frac{q}{4\pi\epsilon_0}\frac{\cos\theta}{s^2+d^2}$; $\cos\theta = \frac{d}{\sqrt{s^2+d^2}}$

$$= \frac{-qd}{4\pi\epsilon_0}\frac{1}{(s^2+d^2)^{3/2}}$$

Electric field due to bound charge $\vec{E}_b$

$$\vec{E}_b = -\frac{\sigma_b}{2\epsilon_0}\hat{z} \qquad \text{because } \hat{n} = -\hat{z}$$

$$\left(\vec{E} = \vec{E}^{(q)} + \vec{E}_b\right)\cdot\hat{z} = E_z$$

$$= \left(-\frac{qd}{4\pi\epsilon_0}\frac{1}{(s^2+d^2)^{3/2}} - \frac{\sigma_b}{2\epsilon_0}\right)$$

$$= \sigma_b / (\chi_e \epsilon_0)$$

This is an equation for σ_b. Solve

$$\boxed{\sigma_b = -\frac{1}{2\pi}\frac{\chi_e}{\chi_e+2}\frac{qd}{(s^2+d^2)^{3/2}}}, \qquad \text{total } q_b = \int \sigma_b \, da$$

$$da = 2\pi s \, ds$$

$$q_b = \frac{-\chi_e}{\chi_e+2} q$$

Reality check
$q \to 0$, get 0
$d \to 0$, " 0
$d \to \infty$, get 0

What about bound charge density $\rho_b = -\vec{\nabla}\cdot\vec{P}$

inside

$$\vec{P} = \frac{\epsilon_0 \chi_e}{\epsilon}\vec{D} \qquad \text{linear dielectric}$$

$$\vec{\nabla}\cdot\vec{P} = \frac{\epsilon_0\chi_e}{\epsilon}\vec{\nabla}\cdot\vec{D} = \frac{\epsilon_0\chi_e}{\epsilon}\underset{0}{\underset{\|}{\rho_{free}}}$$

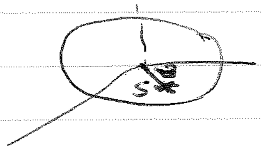
$$\rho_b = 0$$

Force on q ? is due to

σ_b: Can compute resulting $\vec{E}$ at $\vec{r}$ above plane

Compute $\vec{E}$ at position of point charge - 2 ways

1) $$\vec{E}(\vec{r} = d\hat{z}) = \frac{1}{4\pi\epsilon_0} \int \frac{da\, \sigma(s)\, (\vec{r} - \vec{s})}{|\vec{r} - \vec{s}|^3}$$



$$E_z(d\hat{z}) = \frac{1}{4\pi\epsilon_0} \int \frac{da\, \sigma(s)\, d}{(d^2+s^2)^{3/2}}$$

$$da = s\,ds\,d\phi = 2\pi s\,ds$$

$$E_z(d\hat{z}) = \frac{1}{4\pi\epsilon_0} \int_0^\infty \frac{s\,ds}{(d^2+s^2)^{3/2}}\, 2\pi \left(-\frac{\chi_e}{\chi_e+2}\right) \frac{1}{2\pi} \frac{q\,d^2}{(s^2+d^2)^{3/2}}$$

$$= -\frac{1}{4\pi\epsilon_0} \frac{\chi_e}{\chi_e+2}\, q\,d^2 \int_0^\infty \frac{s\,ds}{(s^2+d^2)^3}$$

$$= -\frac{1}{4\pi\epsilon_0} \frac{\chi_e}{\chi_e+2} \frac{q\,d^2}{4\,d^4} = -\frac{1}{4\pi\epsilon_0} \frac{\chi_e}{\chi_e+2} \frac{q}{4d^2}$$

$$\vec{F} = +q\,E_z(d\hat{z})$$

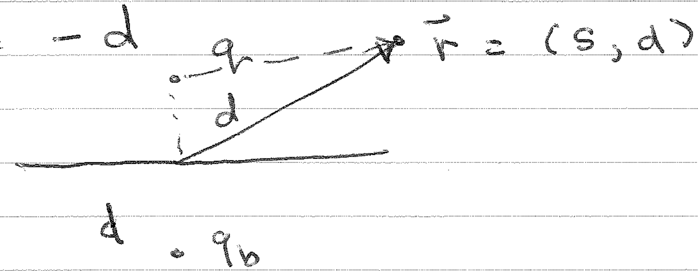
$E_z(d\hat{z}) \propto \frac{1}{(2d)^2}$ suggests

another way - image charge method

Works with $\underline{Z > 0}$

Replace dielectric with image charge q_b at $Z = -d$

Side view



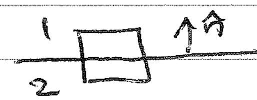
$$V_{>} = \frac{1}{4\pi\epsilon_0}\left[\frac{q}{\sqrt{s^2+(z-d)^2}} + \frac{q_b}{\sqrt{s^2+(z+d)^2}}\right]$$

For $z < 0$ use charge $q + q_b$

$$V_{<} = \frac{1}{4\pi\epsilon_0}\frac{q+q_b}{\sqrt{s^2+(z-d)^2}}$$

$V_{>}$ solves Poisson eq for $z > 0$

$V_{<}$ " Laplace eq for $z < 0$



$(\vec{E}_1 - \vec{E}_2)\cdot\hat{n}$

Need to check boundary condition on $\vec{E} = -\vec{\nabla}V$

$$-\left(\frac{\partial V_{>}}{\partial z} - \frac{\partial V_{<}}{\partial z}\right) = \frac{\sigma}{\epsilon_0} = \frac{\sigma_b}{\epsilon_0}$$

This eqn checks out after doing algebra