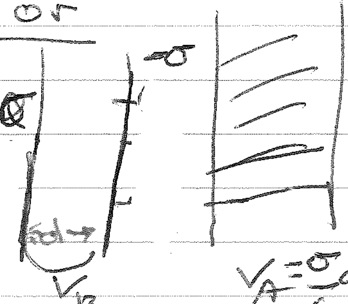
Energy in Capacitor

$\sigma = \frac{Q}{A}$ $\qquad$ $D = \sigma$

ϵ $\qquad$ $E = D/\epsilon_0$

$-\sigma$ $\qquad$ $V_B = \frac{\sigma}{\epsilon_0} d$



$V_A = \frac{\epsilon_0}{\epsilon} V_B$

V_B $\qquad$ $V_A = \frac{\sigma}{\epsilon} d$ $\qquad$ $V_A < V_B$

Capacitor is filled with dielectric. We saw previously that putting dielectric in capacitor reduced the potential difference. This is because the dipoles produce an apposing electric field. Work must be done to maintain the same potential V_B as before. How much work? That is the subject of today's lecture.

Let's compute energy of capacitor W_{cap}. We calculate work done to separate a free charge $+Q$ from $-Q$. If charge dQ is moved from negative plate to positive plate then work done

$$dW_{cap} = V\,dQ = \frac{Q}{C}\,dQ \qquad \frac{Q}{V} = C = \frac{\epsilon A}{d}$$

$$V = \frac{\sigma d}{\epsilon}$$

Total work to transfer charge Q = W_{cap}

$$W_{cap} = \int_0^Q \frac{Q\,dQ}{C} = \frac{Q^2}{2C}$$

How much energy is in the field? W_{field}

$$E = \frac{Q}{\epsilon A}, \quad \text{Volume} = Ad \text{ so}$$

$$W_{field} = \frac{1}{2}\epsilon_0 E^2 (Ad) = \frac{1}{2}\epsilon_0 \frac{Q^2 Ad}{\epsilon^2 A^2} = \frac{\epsilon_0}{2\epsilon}\frac{Q^2}{C}$$

G → "electrostatic energy"

$W_{field} < W_{cap}$ by a factor ϵ_0/ϵ

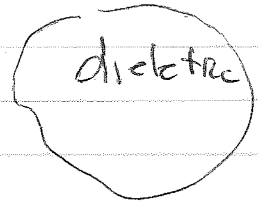
This is because the formula $\frac{\epsilon_0 E^2}{2}$ does not account for the energy needed to polarize dipole.

When dealing with dielectrics we need energy required to build up the charge –

The next step is to generalize the formula for W_{cap}

Energy in dielectric - general treatment

With unpolarized dielectric in place, we bring in free charges, one by one. The dielectric responds



bring in

$\leftarrow q_{free}$

free charge from infinity

ammount of work to bring in free charge Δq

V is potential due to charge density already present

$$\Delta W = \Delta q_f \, V$$

, Δq_f is distributed over some volume

$$= \int (\Delta \rho_f) \, V \, d\tau$$

We controll ρ_f

$$\vec{\nabla} \cdot \vec{D} = \rho_f$$

for an increment $\Delta \rho_f$ we get change in D

$$\vec{\nabla} \cdot (\Delta \vec{D}) = \Delta \rho_f$$

$$\Delta W = \int_{\tau} \vec{\nabla} \cdot (\Delta \vec{D}) \, V \, d\tau$$

3 dim integration by parts

$$\nabla \cdot (\Delta \vec{D} \, V) = \underbrace{(\vec{\nabla} \cdot \Delta \vec{D})}_{= \Delta \rho_f} V + \Delta \vec{D} \cdot \vec{\nabla} V$$

$$= \Delta \rho_f \, V - \Delta \vec{D} \cdot \vec{E}$$

So

$$\Delta W = \int_{v} \left(\vec{\nabla} \cdot (\Delta \vec{D} V) + \Delta \vec{D} \cdot \vec{E} \right) d\tau$$

$$\int_{s} V \Delta \vec{D} \cdot d\vec{a} + \int_{v} (\Delta \vec{D} \cdot \vec{E}) \, d\tau$$

Imagine boundary at ∞

$da \propto r^2$

$D \propto 1/r^2$; $V \sim 1/r$ so

surface integral $\propto \frac{r^2}{r^3}$ with $r \to \infty$

goes to 0

$$\Delta W = \int_{v} (\Delta \vec{D} \cdot \vec{E}) \, d\tau$$

This is general. For a linear dielectric

can go further

$$\vec{D} = \epsilon \vec{E}$$

$$\Delta D = \epsilon \Delta \vec{E}$$

$$\Delta \vec{D} \cdot \vec{E} = \Delta(\epsilon \vec{E}) \cdot \vec{E} = \frac{1}{2} \Delta (\epsilon E^2)$$

$$\Delta W = \Delta \left[\frac{1}{2} \int \vec{D} \cdot \vec{E} \, d\tau \right]$$

$$W = \frac{1}{2} \int \vec{D} \cdot \vec{E} \, d\tau = \boxed{\frac{1}{2} \int \vec{D} \cdot \vec{E} \, d\tau}$$

So for a linear dielectric the energy

density is $\frac{1}{2}\vec{D}\cdot\vec{E}$ $=\frac{1}{2}\epsilon E^2$

not $\frac{\epsilon_0}{2}E^2$ that we had before

The difference is the energy carried

by bound charges. Let's try to

understand this using a concrete model.

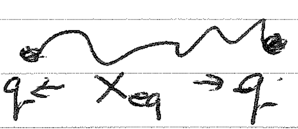
Doing the model right requires quantum

mechanics. Instead do classical mechanics

Consider a model in which electrons & nuclei

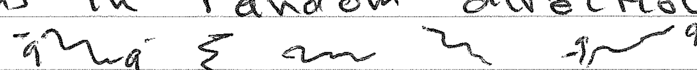
bound by a spring   spring energy $U=\frac{1}{2}kx^2$ $\vec{F}_{spring}=-kx$

$\hat{x}$ →



with no external field

the charges are separated by some equilibrium distance x_{eq}. The system has no dipole moment because there are many atoms with orientations in random directions



Let's say there are n atoms/volume

n = density of atoms

Now turn on electric field in $\hat{x}$-direction

$\vec{E}=E_0\hat{x}$

+q feels force to right

-q feels force to left

The equilibrium position is charged by Δx

so that the total force is balanced

$$\vec{F} = \vec{F}_E + \vec{F}_{spring} = 0$$

$$(qE_0 - k\Delta x) = 0$$

$$\Delta x = \frac{qE_0}{k}$$

Now there is a dipole moment $p_x = \frac{q^2 E_0}{k}$

The polarization is the dipole moment/volume

$$P_x = \frac{q^2 E_0}{k} n$$

$$\boxed{D_x = \epsilon_0 E_0 + P_x} = E_0\left[\epsilon_0 + \frac{nq^2}{k}\right]$$

$$u = \frac{D \cdot E}{2} = \frac{E_0^2}{2}\left(\epsilon_0 + \frac{nq^2}{k}\right)$$

energy density of E field

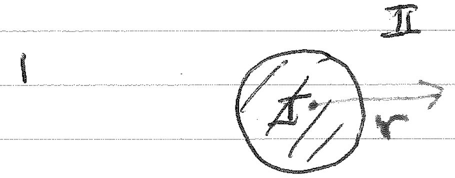
second term $\frac{1}{2} E_0^2 \frac{nq^2}{k} = \boxed{n \frac{1}{2} k^2 \Delta x^2}$

potential energy density of spring

$$u = \frac{D \cdot E}{2} = \epsilon_0 \frac{E^2}{2} + \frac{\vec{P} \cdot \vec{E}}{2}$$

↓ potential energy in spring

Example sphere of radius R filled with dielectric ϵ. There is also a constant density ρ_f of free charge. What is the energy of the charge configuration?



Need $\vec{D}$, $\vec{E}$

Physics is not

(I) Gauss Law make sphere of radius $r < R$

$\vec{D}$ is radial

$$D A = Q_{\text{free enclosed}}$$

$$D \, 4\pi r^2 = \frac{4\pi}{3} \rho_f r^3$$

$$D = \rho_f r / 3$$

$$E = D/\epsilon = \rho_f r / 3\epsilon$$

II

$$D \, 4\pi r^2 = \frac{4\pi}{3} \rho_f R^3$$

$$D = \frac{\rho_f}{3} \frac{R^3}{r^2}$$

$$E = \frac{D}{\epsilon_0} = \frac{\rho_f}{3} \frac{R^3}{r^2} \frac{1}{\epsilon_0}$$

"Electrostatic" energy $\int \frac{1}{2} \epsilon_0 E^2 \, d\tau \equiv W_1$

$d\tau = 4\pi r^2 dr$

$$W_1 = \frac{\epsilon_0}{2} \left[\int_0^R \left(\frac{\rho_f r}{3\epsilon} \right)^2 4\pi r^2 dr + \int_R^\infty \left(\frac{\rho_f}{3\epsilon_0} \frac{R^3}{r^2} \right)^2 4\pi r^2 dr \right]$$

$$= \frac{\epsilon_0 4\pi \rho_f^2}{2 \cdot 9} \left[\frac{R^5}{\epsilon^2 5} + \frac{1}{\epsilon_0^2} R^5 \right] = \frac{2\pi}{9\epsilon_0} \rho_f^2 R^5 \left(\frac{1}{5 \epsilon_r^2} + 1 \right)$$

Total electrostatic energy W_2

energy density $= \frac{1}{2}\vec{D}\cdot\vec{E} = \frac{\epsilon}{2}E^2$ in I

$= \frac{\epsilon_0}{2}E^2$ in II

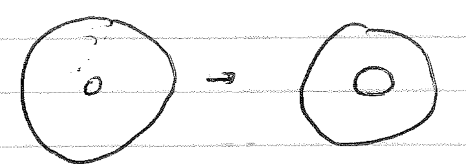
only first integral from 0 to R is changed

$$W_2 = \frac{2\pi \rho_F^2 R^5}{\epsilon_0 9}\left[\frac{1}{\epsilon_r 5} + 1\right]$$

The $W_2 > W_1$ W_2 includes work needed to stretch the molecules

Next check that W_2 is really what we say it is - the energy required to bring in free charge from infinity, Start with <u>uncharged</u> dielectric

Suppose we bring in free charge in increments dq we fill out the sphere layer by layer

 etc

Suppose we reach radius r' $\vec{E}$ is given by $\vec{E}$ in radial direction

$$E(r) = \frac{\rho_f}{3\epsilon_0} r \qquad r < r'$$

$$= \frac{\rho_f}{3\epsilon} \frac{r'^3}{r^2} \qquad r' < r < \infty$$

The work needed to bring in next dq ~~to~~ from ∞ to r' is

$$dW^{(r')} = -dq \left[\int_{\infty}^{R} \vec{E} \cdot d\vec{l} + \int_{R}^{r'} \vec{E} \cdot d\vec{l} \right]$$

$$= -dq \int_{\infty}^{R} \frac{\rho_f}{3\epsilon_0} \frac{r'^3}{r^2} dr - dq \int_{R}^{r'} \frac{\rho_f}{3\epsilon} \frac{r'^3}{r^2} dr$$

$$dW(r') = \left(\frac{\rho_f r'^3}{3\epsilon_0} \left(\frac{1}{R} \right) + \frac{\rho_f r'^3}{3\epsilon} \left[\frac{1}{r'} - \frac{1}{R} \right] \right) dq$$

This increases the radius by dr' $\quad dq = \rho_f 4\pi r'^2 dr'$

Total work $= \int dW(r') = W_2$ see text

$$W_{spring} = W_2 - W_1 = \frac{2\pi}{45\epsilon_0 \epsilon_r^2} \rho_f^2 R^5 (\epsilon_r - 1)$$

Interpret $W_2 - W_1$ in terms of spring model

$$W_2 - W_1 = W_{sp} ??$$

$$= \frac{1}{2} \int d\tau \left(\epsilon E^2 - \epsilon_0 E^2 \right)$$

$$= \frac{1}{2} \int d\tau \, (\epsilon - \epsilon_0) E^2$$

But $\vec{P} = \frac{\epsilon - \epsilon_0}{\epsilon_0} \vec{E}$ so

$$W_2 - W_1 = \frac{1}{2} \int d\tau \, \vec{P} \cdot \vec{E}$$ = spring energy $\frac{1}{2} kx^2$ of previous

$d\vec{p} = \vec{P} \, d\tau$

$$= \frac{1}{2} \int \left(d\vec{p} \right) \cdot \vec{E}$$

$P_0(\cos\theta) = 1$

More mathematical

multiply $V_a = \sum_l \left(A_l r^l + \frac{B_l}{r^{l+1}}\right) P_l(\cos\theta)$

by $P_n(\cos\theta)$, and integrate $d\cos\theta$

LHS has $\int_{-1}^{1} d\cos\theta\, P_n(\cos\theta)\, V_a\, P_0(\cos\theta) \rightarrow V_a \frac{2}{2n+1} \delta_{n0}$

$$\text{RHS} = \frac{2}{2n+1} \delta_{nl} \left(A_l a^l + \frac{B_l}{a^{l+1}}\right)$$

so $A_l a^l + \frac{B_l}{a^{l+1}} = 0$ if $l \neq 0$

Can do same thing at $r = b$

$$A_l b^l + \frac{B_l}{b^{l+1}} = 0$$

Thus if $l \neq 0$ $A_l = 0$, $B_l = 0$

$V(r) = \frac{B}{r}$ drop A constant $B_0 \rightarrow B$

$\vec{E} = -\vec{\nabla} V = \hat{r} \frac{B}{r^2}$ spherically symmetric

radially symmetric Why? $\vec{E}$ depends on total charge bound + free total charge is spherical

check BC on vertical boundaries

tangential $\vec{E}$ continuous $\hat{t} = \hat{r}$ ok

Normal $\vec{D}$ continuous normal $\perp \hat{r}$ so $D_n = 0$

on both sides

Today First Part Boundary Problems with Dielectrics

$\vec{D} = \epsilon_0 \vec{E} + \vec{P}$

interface — 1 ↑n / 2

$(\vec{D}_1 - \vec{D}_2) \cdot \hat{n} = \sigma_f$ General

$(\vec{E}_1 - \vec{E}_2) \cdot \hat{t} = 0$

V is continuos

Surface of conductor is equipotential

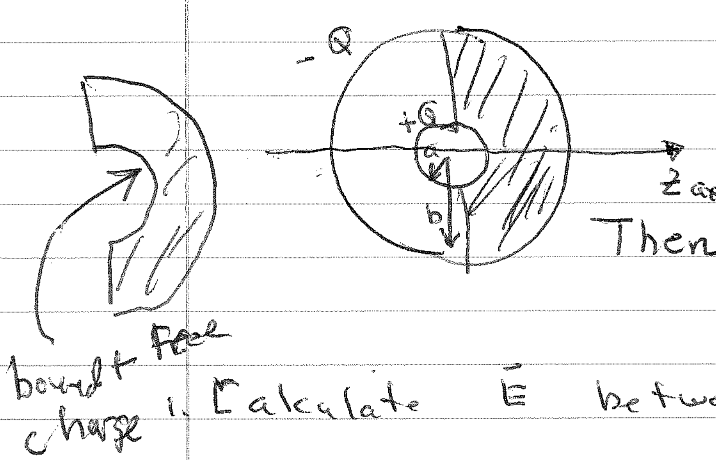
Linear dielectric $\vec{D} = \epsilon \vec{E}$
$= -\epsilon \vec{\nabla} V$

BC on $\vec{D}$:

$$\hat{n} \cdot [-\epsilon_1 \vec{\nabla} V + \epsilon_2 \vec{\nabla} V] = \sigma_f$$

$$-\epsilon_1 \frac{\partial V}{\partial n} + \epsilon_2 \frac{\partial V}{\partial n} = \sigma_f$$

To solve problems all BC must be kept in mind



Two concentric conducting spheres of inner+outer radii a and b, carry free charge +Q, -Q

Then half filled by hemisphere of dielectric ϵ permitivity

1. Calculate $\vec{E}$ between spheres $a < r < b$

z-axis is horizontal to have azimuthal symm.

$$V(r) = \sum_l \left(A_l r^l + \frac{B_l}{r^{l+1}} \right) P_l(\cos\theta)$$

$V(r=a) = V_a =$ const (surface of conductor

$V(r=b) = V_b =$ " "

This means if $l \neq 0$ $A_l = 0, B_l = 0$

only terms with $l=0$ have no angular dependence

To get $\vec{E}$ inside need to get B

In terms of Q

The total free charge on inner surface ($r=a$) is Q

So $\int \vec{D} \cdot da = Q$

$$\int_{\text{RIGHT } \vec{D}=\epsilon\vec{E}} \vec{D} \cdot d\vec{a} + \int_{\text{LEFT } \vec{D}=\epsilon_0\vec{E}} \vec{D} \cdot d\vec{a}$$

$$= \epsilon \frac{B}{a^2} 2\pi a^2 + \epsilon_0 \frac{B\, 2\pi a^2}{a^2} = Q$$

$$B = \frac{Q}{2\pi(\epsilon+\epsilon_0)}$$

$$\boxed{\vec{E} = \frac{Q}{2\pi(\epsilon+\epsilon_0)} \frac{\hat{r}}{r^2}}$$

2. Get surface charge distribution on inner sphere

For $\frac{\pi}{2} \leq \theta \leq \pi$ $\quad \sigma_{free} = \vec{D}_{(a)} \cdot \hat{r} = \epsilon_0 \vec{E}(a) \cdot \hat{r}$

$$= \frac{\epsilon_0 Q}{2\pi(\epsilon+\epsilon_0)} \frac{1}{a^2}$$

For $0 \leq \theta \leq \frac{\pi}{2}$ $\quad \sigma_{free} = \epsilon \vec{E}(a) \cdot \hat{r}$

$$= \frac{\epsilon Q}{2\pi(\epsilon+\epsilon_0) a^2}$$

This is consistent with $\oint \sigma_{free}\, da = Q$

what about bound charge? $\quad \sigma_b = \hat{n} \cdot \vec{P} = \hat{n} \cdot (\epsilon - \epsilon_0) \vec{E}(a)$

$0 \leq \theta \leq \pi/2 \qquad \hat{n} = -\hat{r}$

$$\sigma_B = -(\epsilon - \epsilon_0)\, E(a)$$

$$= -(\epsilon - \epsilon_0) \frac{Q}{2\pi a^2(\epsilon_0 + \epsilon)}$$

note that for $0 \le \theta \le \pi/2$

$$\sigma_{free} + \sigma_B = \frac{\epsilon Q}{2\pi(\epsilon + \epsilon_0)a^2} + \frac{(\epsilon_0 - \epsilon)Q}{2\pi a^2(\epsilon + \epsilon_0)}$$

$$= \frac{\epsilon_0 Q}{2\pi a^2(\epsilon + \epsilon_0)}$$

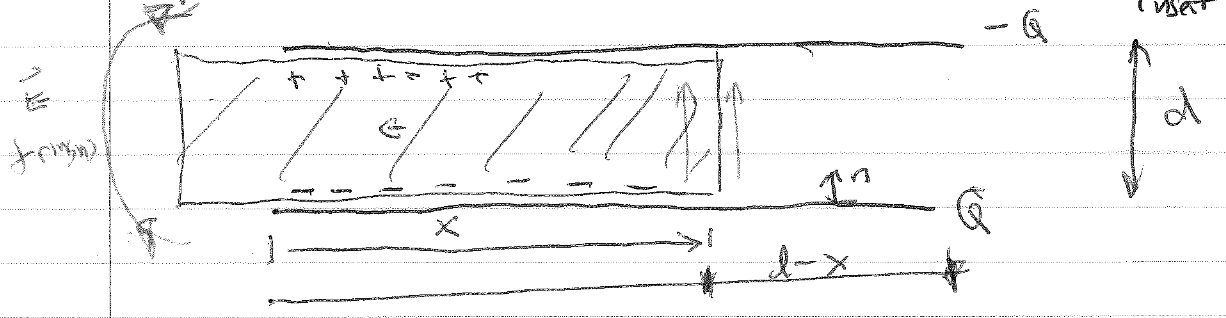
same as for $\frac{\pi}{2} \le \theta \le \pi$

So spherical symm of $\vec{E}$ is justified //

Forces on dielectric

Start with example
Capacitance

Parallel conducting plates of size $l \times w$ (finite)
partially filled with dielectric. Put free charge on then insert dielectric



$d \ll x, l-x$

Let potential difference between plates $= V$

$\vec E$ is the same in the two regions
$\vec E$ $\perp$ to plates is tangent to the boundary
tangential field is continuous. $\vec E$ is same on either side of boundary

$$\vec E = \hat n \frac{V}{d}$$

To compute capacitance relate V to free charge $\pm Q$ on plates

Use Gauss Law to lower plate

$$\int \vec D \cdot da = Q = D\,\text{Area}$$

$D = \epsilon E$ in dielectric Area $= x w$
$D = \epsilon_0 \vec E$ in vacuum Area $= (l-x) w$

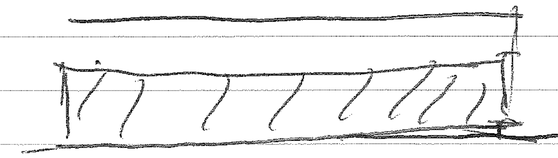
So

$$\epsilon E x w + \epsilon_0 E (l-x) w = Q$$

$$E = V/d, \quad C = Q/V$$

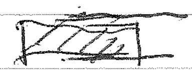
$$C = (\epsilon x + \epsilon_0 (l-x)) \frac{w}{d}$$

Another related example



Whats continuous in the two region
is it D or E or none

Normal components of D are continuous no free charge

Go back to 

Now compute force on dielectric slab

The slab experiences force into capacitor
atoms in dielectric polarized by electric field

But we have always pretended E field is uniform in a parallel plate capacitor - if exactly true no net force to right

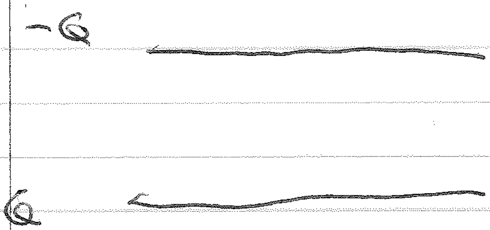
But plates have finite size There is a fringing field - usually small but necessary here

How to calculate force? Direct calculation uses $\vec{E}$ but would need to compute fringing field. Too difficult

Instead use energy consideration

Will determine the force F in two different ways

Plates with fixed charge Isolated



no external wires + Voltage

work done by electrostatic force F if slab moves dx into capacitor is

$F\,dx$. but energy is conserved have field energy W total energy is conserved

$$F\,dx + dW = 0$$

$$F = -\frac{dW}{dx}$$

charge Q is fixed use

Here Q is fixed $W = Q^2/2C$

$$W = \frac{Q^2 d}{2w[\epsilon x + \epsilon_0(l-x)]}$$

$\epsilon > \epsilon_0$ as x increases W decreases

Force on slab is

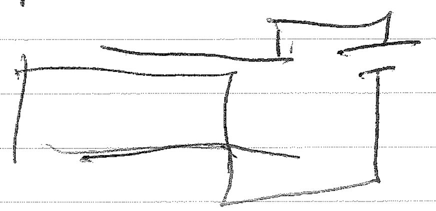
$$F = -\frac{dW}{dx} = \frac{Q^2 d}{2w}\frac{(\epsilon-\epsilon_0)}{[\epsilon x + \epsilon_0(l-x)]^2} = \frac{V^2 w(\epsilon-\epsilon_0)}{2d}$$

using $C = QV$

For fixed Q the force decreases as x increases

Another example - plates with fixed potential difference

connected to



Battery plates

now work done by electrostatic force F is
not $-dW$ because additional energy
is supplied by battery

If slab moves distance dx then dQ
is transferred to plates from ~~battery~~
battery supplies dQV

$$\boxed{\text{now}}\ dW = V\,dQ - F\,dx$$

$$F\,dx = -dW + V\,dQ$$

$$dW + F\,dx = V\,dQ$$

V is constant
C changes with x

now use $W = \frac{1}{2} C V^2$

V is fixed

$$Q = CV, \qquad dQ = dC\,V$$

$$F\,dx = dC\,V^2 - dQ\,V$$

$$= dC\,V^2 - \frac{1}{2} dC\,V^2 = \frac{dC}{2} V^2$$

$$F = \frac{1}{2}\frac{dC}{dx} V^2$$

$$= \frac{V^2}{2d} w (\epsilon - \epsilon_0)$$

same as before

If V is constant F is independent of x

We have neglected end (fringing) effects & used $d \ll l$, $d \ll x$

if $x = l$ $F = 0$ so formula is NG