Energy in capacitor

Capacitor is filled with dielectric. We saw previously that putting dielectric in capacitor reduced the potential difference.

This is because the dipoles produce an opposing electric field.

Work must be done to maintain the same potential as before.

How much work? That is the subject of today's lecture.

Let's compute energy of capacitor

We calculate work done to separate a free charge $+Q$ from $-Q$.

If the charge is moved from negative plate to positive plate, then work done:

$$dW = \frac{Vd}{C} \cdot Q = Q \cdot d \cdot \frac{Q}{V} = C = \frac{\varepsilon A}{d}$$

$$V = \frac{\varepsilon d}{e}$$
Total work to transfer charge $\Delta Q = W_{\text{cap}}$

$$W_{\text{cap}} = \int \frac{q dQ}{\varepsilon_0} = \frac{Q^2}{2\varepsilon_0}$$

How much energy is in the field? $W_{\text{field}}$

$$E = \frac{Q}{\varepsilon_0 A} \quad \text{Volume} = Ad \quad \text{so}$$

$$W_{\text{field}} = \frac{1}{2} \varepsilon_0 E^2 = \frac{1}{2} \varepsilon_0 \frac{Q^2 Ad}{\varepsilon^2 A^2} = \frac{\varepsilon_0}{2} \frac{Q^2}{C}$$

$W_{\text{field}} < W_{\text{cap}}$ by a factor $\frac{\varepsilon_0}{\varepsilon}$

This is because the formula $\frac{\varepsilon_0 E^2}{2}$ does not account for the energy needed to polarize the dipole.

When dealing with dielectrics, we need energy required to build up the charge.

The next step is to generalize the formula for $W_{\text{cap}}$. 
Energy in dielectric - general treatment

With un polarized dielectric in place, we bring in free charges one by one. The dielectric responds

\[ \Delta W = \Delta q_f V \]

\[ = \int (\Delta P_f) \cdot V \, dV \]

We control \( P_f \)

\[ \nabla \cdot \vec{D} = \rho_f \]

\[ \nabla \cdot (\Delta \vec{D}) = \Delta \rho_f \]

\[ \Delta W = \int \nabla \cdot (\Delta \vec{D}) \cdot V \, dV \]

3 dim integration by parts

\[ \nabla \cdot (\Delta \vec{D} V) = \left( \nabla \cdot \Delta \vec{D} \right) V + \Delta \vec{D} \cdot \nabla V \]

\[ = \Delta P_f - \Delta \vec{D} \cdot \vec{E} \]

So
\[ \Delta W = \int \left( \nabla \cdot (\Delta \mathbf{D} \mathbf{V}) + \Delta \mathbf{D} \cdot \mathbf{E} \right) \, d\tau \]

Imagine boundary at \( \infty \)
\[ d\sigma \propto \frac{1}{r^2} \]
\[ D \propto \frac{1}{r^2} \quad \Rightarrow \quad V \propto \frac{1}{r} \quad \text{so} \]

Surface integral \( \propto \frac{1}{r^3} \) with \( \lim_{r \to \infty} \]

goes to 0

\[ \Delta W = \int \Delta \mathbf{D} \cdot \mathbf{E} \, d\tau \]

This is general. For a linear dielectric

\[ \Delta \mathbf{D} = \varepsilon \Delta \mathbf{E} \]
\[ \Delta \mathbf{D} \cdot \mathbf{E} = \Delta (\varepsilon \mathbf{E}) \cdot \mathbf{E} = \frac{1}{2} \Delta (\varepsilon \mathbf{E}^2) \]

\[ \Delta W = \Delta \left( \frac{1}{2} \int \Delta (\varepsilon \mathbf{E}^2) \, d\tau \right) \]

\[ W = \frac{1}{2} \int \Delta \mathbf{D} \cdot \mathbf{E} \, d\tau = \frac{1}{2} \int \Delta (\varepsilon \mathbf{E}^2) \, d\tau \]
So for a linear dielectric the energy density is \( \frac{1}{2} \varepsilon E^2 \) instead of \( \frac{1}{2} \varepsilon_0 E^2 \) that we had before.

The difference is the energy carried by bound charges. Let's try to understand this using a concrete model.

Doing the model right requires quantum mechanics. Instead, do classical mechanics.

Consider a model in which electrons are nuclei bound by a spring. Spring energy \( U = \frac{1}{2} k x^2 \).

The charges are separated by some equilibrium distance \( x_{eq} \). The system has no dipole moment because there are many atoms with orientations in random directions.

Let's say there are \( n \) atoms/volume.

Now turn on an electric field in \( z \)-direction.

\[ E = E_0 z \]

\( +q \) feels force to right,
\( -q \) feels force to left.

\[ F = qE \]

\( x \rightarrow x_{eq} \rightarrow x \) with no external field.
The equilibrium position is changed by \( \Delta x \) so that the total force is balanced

\[
\vec{F} = \vec{F}_E + \vec{F}_{\text{spring}} = 0
\]

\[
(q \vec{E}_0 - \hbar \Delta \vec{x}) = 0
\]

\[
\Delta x = \frac{q \vec{E}_0}{\hbar}
\]

Now there is a dipole moment \( \vec{P}_x = \frac{q^2 \vec{E}_0}{\hbar} \).

The polarization is the dipole moment / volume

\[
\vec{P}_x = \frac{q^2 \vec{E}_0}{\hbar}
\]

\[
\quad
\]

\[
\vec{D}_x = \varepsilon_0 \vec{E}_0 + \vec{P}_x = \varepsilon_0 \left[ \varepsilon_0 + \frac{n q^2}{\hbar^2} \right]
\]

\[
\vec{u} = \frac{\vec{D} \cdot \vec{E}}{2} = \frac{\varepsilon_0^2}{2} \left( \varepsilon_0 + \frac{n q^2}{\hbar} \right)
\]

Energy density of \( \vec{E} \) field

Second term \( \frac{1}{2} \varepsilon_0^2 \frac{n q^2}{\hbar} = \frac{n}{2} \hbar^2 \Delta x^2 \)

Potential energy density of spring

\[
\vec{u} = \frac{\vec{D} \cdot \vec{E}}{2} = \varepsilon_0 \frac{E^2}{2} + \frac{\vec{P} \cdot \vec{E}}{2}
\]

Potential energy in spring
Example: Sphere of radius R filled with dielectric $\varepsilon$. What is the energy of the charge configuration?

I. Need $D$, $E$

\[ D = \frac{P_f \, R^3}{3} \]

\[ E = \frac{D}{\varepsilon} = \frac{P_f \, R^3}{3\varepsilon} \]

II. Gauss Law: Make sphere of radius $r < R$

\[ D \cdot 4\pi r^2 = \frac{4\pi}{2} P_f \, r^3 \]

\[ D = \frac{P_f \, r^3}{3} \]

\[ E = \frac{1}{2} \int \frac{1}{\varepsilon_0} \, \hat{r} \cdot \mathbf{E} \, dV = \frac{1}{2} \int \frac{1}{\varepsilon_0} \, \hat{r} \cdot \mathbf{E} \, dV \]

Electric potential energy:

\[ W_1 = \frac{\varepsilon_0}{2} \int \int \int \left( \frac{P_f \, r^3}{3\varepsilon} \right) 4\pi r^2 \, dr + \int \int \int \left( \frac{P_f \, r^3}{3\varepsilon_0} \right) 4\pi r^2 \, dr \]

\[ = \frac{\varepsilon_0 \, 4\pi \, P_f^2}{2} \left[ \frac{R^5}{5} + \frac{1}{\varepsilon_0} \frac{R^5}{6} \right] = \frac{2\pi \, P_f^2 \, R^5}{9\varepsilon_0} \left( \frac{1}{\varepsilon_0^2} + 1 \right) \]
Total electrostatic energy \( W_2 \)

Energy density

\[
\frac{1}{2} \int E \cdot E = \frac{1}{2} \varepsilon \varepsilon_0 E^2 \quad \text{in I}
\]

\[
\frac{1}{2} \varepsilon \varepsilon_0 E^2 \quad \text{in II}
\]

Only first integral \( r \) from \( 0 \) to \( R \) is changed

\[
W_2 = \frac{2 \pi \varepsilon_0 \varepsilon_r^2 R^5}{39} \left[ \frac{1}{\varepsilon_r^5} + 1 \right]
\]

The \( W_2 > W_1 \) \( W_2 \) includes work needed to stretch the molecules.

Next check that \( W_2 \) is really what we say it is - the energy required to bring in free charge from infinity. Start with uncharged dielectric

Suppose we bring in free charge in increments \( dq \) we fill out the sphere layer by layer

\( 0 \to 0 \) etc
Suppose we reach radius \( r^1 \). If is swollen in radial direction,

\[
E(r) = \begin{cases} \frac{Pf}{3\varepsilon} r & r \leq r^1 \\ \frac{Pf}{3\varepsilon} \frac{r^1^3}{r^2} & r^1 < r < \infty \end{cases}
\]

The work needed to bring in next dq to from \( \infty \) to \( r^1 \) is

\[
dW(r) = dq \left[ \int_\infty^R E\, dr + \int_{r^1}^{R} E\, dr \right]
\]

\[
= -dq \left[ \int_\infty^{r^1} \frac{Pf}{3\varepsilon} \frac{r^{1^3}}{r^2} \, dr - dq \int_{r^1}^{R} \frac{Pf}{3\varepsilon} \frac{r^{1^3}}{r^2} \, dr \right]
\]

\[
dW(r) = \left( \frac{Pf}{3\varepsilon} \left( \frac{1}{R} \right) + \frac{Pf}{3\varepsilon} \left[ \frac{1}{r^1} - \frac{1}{R} \right] \right) dq
\]

This increases two radius, \( dq = Pf 4\pi r^2 \, dr \)

Total \( \delta W = \int dW(r) = W_2 \) sectar +

\[
W_{spring} = W_2 - W_1 = \frac{2\pi}{45\varepsilon_0 E_r^2} Pf R^5 (E_r - 1)
\]
Interpret $W_2 - W_1$ in terms of a spring model

$$W_2 - W_1 = W_{sp} z^2$$

$$= \frac{1}{2} \int d\zeta \left( \varepsilon E^2 - \varepsilon_0 \varepsilon^2 \right)$$

$$= \frac{1}{2} \int d\zeta \left( \varepsilon - \varepsilon_0 \right) \varepsilon^2$$

But $\bar{P} = \varepsilon - \varepsilon_0 \varepsilon$ so

$$W_2 - W_1 = \frac{1}{2} \int d\zeta \bar{P} \varepsilon = \text{spring energy} \cdot \frac{1}{2} \varepsilon_0 x^2$$

$$d\bar{P} = \bar{P} d\zeta$$

$$= \frac{1}{2} \int (d\bar{P}) \varepsilon$$
More mathematical
multiply \( V_a = \sum_{l} \left( A_l r^l + B_l \frac{1}{r^{l+1}} \right) P_l(\cos \theta) \)
by \( P_n(\cos \theta) \) and integrate \( d\cos \theta \)
LHS has \( \int d\cos \theta \ P_n(\cos \theta) V_a P_l(\cos \theta) = V_a \frac{2 \delta_{nl}}{2n+1} \)
RHS = \( \frac{2}{2n+1} \delta_{nl} \left( A_l r^l + B_l \frac{1}{r^{l+1}} \right) \)
so \( A_l r^l + B_l \frac{1}{r^{l+1}} = 0 \) if \( l \neq 0 \)

Can do same thing at \( r = b \)
\( A e^b b^l + B e^{b} \frac{1}{b^{l+1}} = 0 \)

Thus if \( l \neq 0 \) \( A e^b = 0, \ B e^{b} = 0 \)

\( V(r) = \frac{B}{r} \) drop \( A \) constant \( B_0 \rightarrow B \)

\( \bar{E} = \nabla \bar{V} = \frac{B}{r^2} \) Spherically symmetric
Radially symmetric why? \( E \) depends on total charge \( \pm \) bound free total charge is spherically

Check BC on vertical boundaries
tangential \( \mathbf{E} \) continuous \( \mathbf{E} = \hat{r} \) ok
Normal \( \mathbf{B} \) continuous normal \( \mathbf{B} \) so \( \mathbf{n} \) \( = 0 \) on both sides
Today First Part Boundary Problems with Dielectrics
\[ \vec{D} = \varepsilon_0 \vec{E} + \vec{P} \]

Interface
\[ \frac{1}{\varepsilon_1} \frac{1}{\varepsilon_2} n \]

\[ (\vec{P}_i - \vec{P}_o) \cdot \hat{n} = \sigma_f \]
\[ (\varepsilon_1 \vec{E}_i - \varepsilon_2 \vec{E}_o) \cdot \hat{n} = 0 \]
\[ V \text{ is continuous} \]

Linear dielectric \[ \vec{D} = \varepsilon \vec{E} \]
\[ \vec{E} = -\varepsilon \nabla V \]

Boundary Condition
\[ \hat{n} \cdot \left[ -\varepsilon_1 \nabla V + \varepsilon_2 \nabla V \right] = \sigma_f \]
\[ -\frac{\varepsilon_1}{dn} \frac{dV}{dn} + \frac{\varepsilon_2}{dn} \frac{dV}{dn} = \sigma_f \]

To solve problems all must be kept in mind

Two concentric conducting spheres of inner radius \( a \) and \( b \), inner charge \( -Q \)

Then half filled by hemisphere of dielectric \( \varepsilon \) pointing

Calculate \( E \) between spheres \( a < r < b \)

Z-axis is horizontal to have \( \theta \) in radian

\[ V(r) = \sum_{n=0}^{\infty} \left( \frac{A_n r^n + B_n}{r^{n+1}} \right) \text{P} \cos(n \theta) \]

\[ V(r=a) = V_a = \text{const (surface of conductor)} \]
\[ V(r=b) = V_0 = " " \]
This means if \( l=0 \)
\[ A_l = 0, B_l = 0 \]

Only terms with \( l=0 \) have no angular dependence.
To get $E$ inside need to get $B$

Interms of $Q$

The total free charge on inner surface is $Q$

\[
\int \mathbf{D} \cdot d\mathbf{a} = Q
\]

\[
\int \mathbf{D} \cdot d\mathbf{a} = \int \mathbf{B} \cdot d\mathbf{a}
\]

\[
\text{RIGHT } \mathbf{D} = 0 \quad \text{LEFT } \mathbf{D} = \varepsilon_0 \mathbf{E}
\]

\[
= \varepsilon \frac{\mathbf{E} \cdot 2\pi a^2}{a^2} + \varepsilon_0 \frac{B \cdot 2\pi a^2}{a^2} = Q
\]

\[
B = \frac{Q}{2\pi (\varepsilon + \varepsilon_0)}
\]

\[
\mathbf{E} = \frac{Q}{2\pi (\varepsilon + \varepsilon_0)} \frac{\hat{r}}{r^2}
\]

2. Get surface charge distribution on inner sphere

For $\frac{\pi}{2} \leq \theta \leq \pi$

\[
\sigma_{\text{free}} = \frac{\mathbf{D} \cdot \hat{r}}{(a)} = \varepsilon_0 \mathbf{E}(a) \cdot \hat{r}
\]

\[
= \frac{\varepsilon_0 Q}{2\pi (\varepsilon + \varepsilon_0) a^2}
\]

For $0 \leq \theta \leq \frac{\pi}{2}$

\[
\sigma_{\text{free}} = \frac{\mathbf{E} \cdot \hat{r}}{(a)}
\]

\[
= \frac{\varepsilon Q}{2\pi (\varepsilon + \varepsilon_0) a^2}
\]

This is consistent with $\int \sigma_{\text{free}} d\mathbf{a} = Q$

What about bound charge? $\sigma_B = \hat{n} \cdot \mathbf{P} = \hat{n} \cdot \varepsilon_0 \mathbf{E}(a)$

For $0 \leq \theta \leq \pi/2$

$\hat{n} \cdot \hat{r} = 1$
\[ \sigma_B = -(\varepsilon - \varepsilon_0) \varepsilon(\omega) \]

\[ = - (\varepsilon - \varepsilon_0) \frac{G}{2\pi a^2 \varepsilon_0^2} \]

Note that for \( 0 \leq \theta \leq \pi/3 \)

\[ G_{\text{rec}} + \sigma_B = \frac{\varepsilon_0 G}{2\pi \varepsilon_0^2 a^2} + \frac{(\varepsilon - \varepsilon_0) G}{2\pi \varepsilon_0 (\varepsilon - \varepsilon_0)} \]

\[ = \frac{\varepsilon_0 G}{2\pi a^2 (\varepsilon - \varepsilon_0)} \]

Same as for \( \pi < \theta \leq \pi \)

So spherical symmetry \( \varepsilon \) is justified
Forces on dielectric
Parallel conducting plates of size \( l \times w \) (finite)
partially filled with dielectric
Put free charge on then
Insert dielectric

Let potential difference between plates \( V \)

\( E \) is the same in the two regions
\( E \) to plates is tangent to the boundary
tangential field is continuous. \( E \) is same
on either side of boundary

\[
E = \frac{\nabla V}{d}
\]

To compute capacitance relate \( V \) to free charge
\( \pm Q \) on plates

Use Gauss's Law to lower plate

\[
\int D \, dA = \int Q = \text{Area} \\
D = \varepsilon E \text{ in dielectric} \quad \text{Area} = x \, w \\
D = \varepsilon_0 E \text{ in vacuum} \quad \text{Area} = (l-x) \, w
\]

So
\[
\varepsilon E \, x \, w + \varepsilon_0 E \, (l-x) \, w = Q
\]

\[
E = \frac{V}{d} \quad C = \frac{Q}{V}
\]

\[
C = \frac{(\varepsilon \, x + \varepsilon_0 (l-x)) \, w}{d}
\]
Another related example

What is continuous in the two regions is it D or E or none

Normal component of D are continuous no free charge

Go back to

Now compute force on dielectric slab

The slab experiences force into capacitor atoms in dielectric polarized by electric field

But we have always presumed E-field is uniform in a parallel plate capacitor - if exactly true no net force to right

But plates have finite size. There is a fringing field - usually small but necessary here

How to calculate force? Direct calculation would be used to compute fringing field, too difficult

Instead use energy considerations
will determine the force $F$ in two different ways

Plates with fixed charge \( \frac{Q}{2} \) \( \text{Isolated} \) \( \text{no external wires \& Voltage} \)

Work done by electrostatic force $F$
If slab moves, $dx$ into capacitor is

$F dx \cdot \text{but energy is conserved into field energy } W$
Total energy is conserved

$F dx + dW = 0$
$F = - \frac{dW}{dx}$ \( \text{Charge is fixed-use} \)

Here $Q$ is fixed $W = \frac{Q^2}{2 \epsilon C}$

$W = \frac{Q^2 d}{2 \epsilon [\epsilon x + \epsilon_0 (l-x)]}$ \( \epsilon \gg \epsilon_0 \)

$W$ decreases as $x$ increases

Force on slab is

$F = - \frac{dW}{dx} = \frac{Q^2 d}{2 \epsilon} \frac{\epsilon - \epsilon_0}{[\epsilon x + \epsilon_0 (l-x)]^2} = \frac{V^2}{2} \epsilon (\epsilon - \epsilon_0)$

Using $\epsilon = \epsilon_0$

For fixed $Q$ the force decreases as $x$ increases
Another example - Plates with fixed potential difference connected to battery.

Now work down by electrostatic force $F$ is

$\text{ho} - dW$ because additional energy is supplied by battery.

If slab moves distance $dx$ then $dQ$ is transferred to plates from battery.

$\Delta V = \frac{1}{2} \epsilon_0 V^2$ as $V$ is constant.

Use $Q = CV$, $dQ = dCV$.

$Fdx = dC V^2 - dQV = dC V^2 - \frac{1}{2} dC V^2 = \frac{dC V^2}{2}$

$F = \frac{1}{2} \frac{dC V^2}{dx^2}$

$\frac{V^2}{2x} (\varepsilon - \varepsilon_0) \quad \text{simplified below}$

If $V$ is constant $F$ is independent of $x$.

We have neglected end (fringing) effects. Used $dx = \frac{dx}{x}$.

If $x = l$, $F = 0$, so formula is $\varepsilon_0$. 
