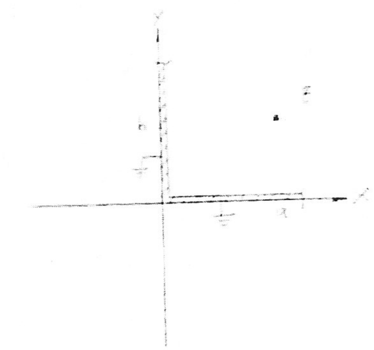
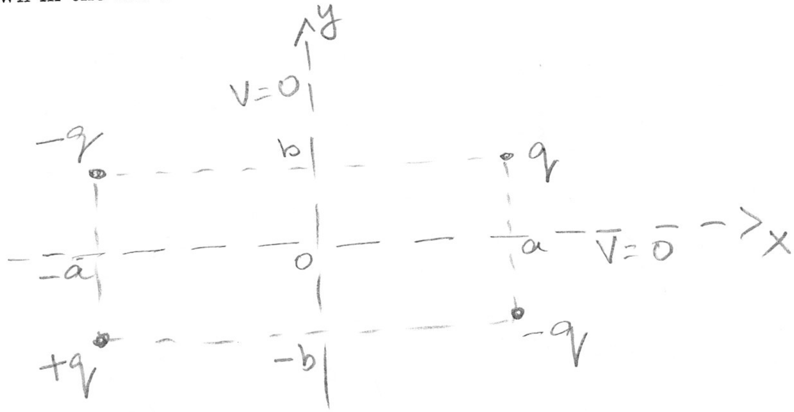
NAME: Student ID: Score:

1. (25 pts total) *Method of Images*
Two semi-infinite grounded conducting planes meet at right angles. In the region between them, there's a point charge q situated as shown in the left sketch below.





(a) (10 pts) To the right of the sketch above, make a careful sketch of the corresponding image-charge problem. Indicate the position(s) and value(s) of the image charge(s).

(b) (7 pts) In terms of variables and constants given in the problem, find the potential in the region $(x > 0, y > 0)$.

Potential is that from the 1 real charge and 3 image charges.

$$V(x,y) = \frac{q}{4\pi\epsilon_0}\left[\frac{1}{\sqrt{(x-a)^2+(y-b)^2}} - \frac{1}{\sqrt{(x+a)^2+(y-b)^2}} - \frac{1}{\sqrt{(x-a)^2+(y+b)^2}} + \frac{1}{\sqrt{(x+a)^2+(y+b)^2}}\right]$$

$(x>0, y>0)$

(c) (8 pts) In terms of variables and constants given in the problem, find the work done to bring the charge to its position from infinity.

First bring charge in from (1) (∞, ∞) to (a, ∞) along line parallel to x-axis then (2) (a,∞) to (a,b) along line parallel to y-axis (line $x=a$). [PATH (1) then PATH (2)]. $W = \int \vec{F} \cdot d\vec{l}$

Drop $(1/4\pi\epsilon_0)$ for compactness:

Force needed to oppose electric force at (x,y) $= -q\left[-\frac{q}{(2y)^2}\hat{j} - \frac{q}{(2x)^2}\hat{i} + \frac{q}{4(x^2+y^2)}\frac{x\hat{i}+y\hat{j}}{\sqrt{x^2+y^2}}\right].$

PATH (1): $W_1 = +\int_\infty^a \frac{q^2}{4x^2}dx = -\frac{q^2}{4a}$

PATH (2): only component along $\hat{j}$ contributes

$$W_2 = +\int_\infty^b \frac{q^2}{4y^2}dy - \int_\infty^b \frac{q^2\, y\, dy}{4(a^2+y^2)^{3/2}} = -\frac{q^2}{4b} + \frac{q^2}{4}\cdot\frac{1}{\sqrt{a^2+b^2}}$$

Total Work $= \left[-\frac{q^2}{4a} - \frac{q^2}{4b} + \frac{q^2}{4}\frac{1}{\sqrt{a^2+b^2}}\right] \times \frac{1}{4\pi\epsilon_0}$

NAME: Student ID: Score:

2. (25 pts total) *Spherical shell with non-uniform potential*
A spherical shell of radius R has a non-uniform (but azimuthally symmetric) electric potential $V_0(\theta) = k\cos^2(\theta)$, where k is a constant and θ is the usual polar angle in spherical coordinates.
(a) (5 pts) What is the general solution $V(r, \theta, \phi)$ to Laplace's equation for the region outside the sphere, before applying boundary conditions at $r = R$? Take the reference potential to be zero at infinity and simplify as much as possible.

$$V(r,\theta) = \sum_{l=0}^{\infty}\left(\underbrace{A_l r^l}_{0} + \frac{B_l}{r^{l+1}}\right)P_l(\cos\theta) = \sum_{l=0}^{\infty}\frac{B_l}{r^{l+1}}P_l(\cos\theta)$$

$A_l r^l$ blows up as $r\to\infty$.

(b) (8 pts) Express the surface potential $V_0(\theta)$ in terms of Legendre polynomials.

$$V_0(\theta) = k\cos^2\theta$$

$$P_2(\cos\theta) = \frac{3\cos^2\theta - 1}{2}.$$

$$\Rightarrow \cos^2\theta = \frac{2P_2(\cos\theta)+1}{3}$$

$$= \frac{2}{3}kP_2(\cos\theta) + \frac{k}{3}$$

$$= \frac{2}{3}P_2(\cos\theta) + \frac{1}{3}$$

$$V_0(\theta) = \frac{2k}{3}P_2(\cos\theta) + \frac{k}{3}P_0(\cos\theta)$$

(c) (8 pts) Apply the boundary condition at the sphere surface and solve for V outside the sphere.

Apply boundary condition: $V(R,\theta) = V_0(\theta)$

$$\sum_{l=0}^{\infty}\frac{B_l}{R^{l+1}}P_l(\cos\theta) = \frac{2k}{3}P_2(\cos\theta) + \frac{k}{3}P_0(\cos\theta)$$

Thus $\frac{B_2}{R^3} = \frac{2k}{3} \Rightarrow B_2 = \frac{2kR^3}{3}$ and $\frac{B_0}{R} = \frac{k}{3} \Rightarrow B_0 = \frac{kR}{3}$.

all other $B_l = 0$. $V_{outside} = V(r,\theta) = \frac{kR}{3}\frac{1}{r} + \frac{2kR^3}{3}\frac{1}{r^3}\left(\frac{3\cos^2\theta - 1}{2}\right)$

(d) (4 pts) What is the electric dipole moment of the sphere around its center?

The potential outside the sphere has no $\frac{1}{r^2}$ component.
Thus $\vec{p} = 0$. Electric dipole moment about center is zero.

NAME: Student ID: Score:

3. (25 pts total) *Spherical shell with non-uniform surface charge*
The surface-charge density of a spherical shell of radius R is specified to be $\sigma(\theta) = \sigma_0 \cos(\theta)$ where σ_0 is a constant and θ is the usual polar angle in spherical coordinates.
(a) (5 pts) What is the total charge on the shell? Write down the integral and evaluate it.

Total charge $Q = \int_0^{\pi}\int_0^{2\pi}\int_0^{R} \sigma_0 \cos\theta \, \delta(r-R)\, r^2 \sin\theta \, dr\, d\phi\, d\theta$

$= \sigma_0 R^2 \int_0^{\pi}\int_0^{2\pi} \cos\theta \sin\theta \, d\phi\, d\theta = \sigma_0 R^2 2\pi \int_0^{\pi} \cos\theta \sin\theta \, d\theta.$

(sub $x = \cos\theta$)

$= \sigma_0 R^2 2\pi \int_1^{-1} x(-dx) = \sigma_0 R^2 2\pi \left[\frac{1}{2}x^2\right]_{-1}^{1} = 0 \Rightarrow \underline{Q=0}$

(b) (9 pts) Find the electric dipole moment of the charge distribution around the sphere center.

$\vec{p} = \int \vec{r}' \rho(\vec{r}') d\tau' = \int (x\hat{i} + y\hat{j} + z\hat{k}) \sigma_0 \cos\theta \, \delta(r-R) r^2 \sin\theta \, dr\, d\phi\, d\theta.$

Recognize that $\vec{p}$ has no $\hat{i}$ or $\hat{j}$ component. This is because $\sigma(x,y,z) = \sigma(-x,y,z) = \sigma(x,-y,z)$. Thus:

$\vec{p} = \hat{k} \int_0^{\pi}\int_0^{2\pi}\int_0^{R} r\cos\theta \, \sigma_0 \cos\theta \, \delta(r-R)\, r^2 \sin\theta \, dr\, d\phi\, d\theta = \hat{k}\sigma_0 R^3 2\pi \int_0^{\pi} \cos^2\theta \sin\theta \, d\theta$

$(z = r\cos\theta)$

$= \hat{k}\sigma_0 R^3 2\pi \int_{-1}^{1} x^2 dx = \hat{k}\sigma_0 R^3 2\pi \frac{1}{3}(1+1) = \boxed{\hat{k}\frac{4\pi R^3 \sigma_0}{3} = \vec{p}}$

(c) (3 pts) For $r > R$, what is the value of the potential on the mid-plane defined by $\theta = \pi/2$? Explain.

The charge distribution $\sigma = \sigma_0 \cos\theta$ has the property $\sigma(x,y,z) = -\sigma(x,y,-z)$. Thus the contribution from the upper and lower parts cancel each other for any (x,y) on the $\theta = \pi/2$ plane. So, $V = 0$ on the $\theta = \pi/2$ plane (xy plane).

(d) (8 pts) Determine the potential for $r \gg R$ to leading order in $(1/r)^n$.

Monopole term ie $\frac{1}{r}$ term is zero since $Q = 0$ (from (a))

Dipole term is dominant for $r \gg R$, since $\vec{p} \neq 0$ (from (b))

Thus for $r \gg R$, $V(r,\theta) = \frac{1}{4\pi\epsilon_0}\frac{\vec{p}\cdot\hat{r}}{r^2} = \frac{1}{4\pi\epsilon_0}\frac{\frac{4\pi}{3}R^3\sigma_0 \cos\theta}{r^2}$

or, $V(r,\theta) = \frac{R^3 \sigma_0}{3\epsilon_0}\frac{\cos\theta}{r^2}$ to leading order in $\left(\frac{1}{r}\right)^n$.