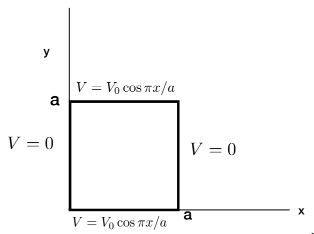
PHYSICS 321:
ELECTROMAGNETISM

21 Nov. 2019 Midterm 2 Solutions

1. *Laplace's equation* A very, very long square conducting pipe of side length a lies along the z axis. The boundary conditions for $V(x,y)$ are $V(0,y)=0$, $V(a,y)=0$, $V(x,0)=V_0\cos\pi x/a$, $V(x,a)=V(x,0)=V_0\cos\pi x/a$.



(a) (6) Explain why the function $f_n(x,y)\equiv\sin\frac{n\pi x}{a}\cosh[\frac{n\pi}{a}(y-a/2)]$, where n is an integer ≥ 1 is a solution of Laplace's equation.

Solving Laplace's equation for a two-dimensional function requires that $[\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}]f_n(x,y)=0$. Here $\frac{\partial^2}{\partial x^2}\sin\frac{n\pi x}{a}=-(\frac{n\pi}{a})^2\sin\frac{n\pi x}{a}$ and $\frac{\partial^2}{\partial y^2}\cosh[\frac{n\pi}{a}(y-a/2)]=+(\frac{n\pi}{a})^2\cosh[\frac{n\pi}{a}(y-a/2)]$, so the sum of the two second partial derivatives vanishes. So Laplace's equation is satisfied.

(b) (4) Explain why the function $f_n(x,y)$ satisfies the boundary conditions at $x=0$ and $x=a$.

The function $\sin\frac{n\pi x}{a}$ vanishes at $x=0$ because $\sin 0=0$, and at $x=a$ because $\sin n\pi=0$ if n is an integer.

(c) (10) Determine $V(x,y)$. You may express your answer in terms of *well-defined*, one-dimensional integrals.

Try $V(x,y)=\sum_n C_n\sin\frac{n\pi x}{a}\cosh[\frac{n\pi}{a}(y-a/2)]$. Evaluate at $y=a$ to get
$V_0\cos\frac{\pi x}{a}=\sum_n C_n\sin\frac{n\pi x}{a}\cosh\frac{n\pi}{2}$. The same equation is obtained for $y=0$. This is a Fourier transform equation. Thus
$C_n=\frac{2V_0}{a\cosh\frac{n\pi}{2}}\int_0^a\cos\frac{\pi x}{a}\sin\frac{n\pi x}{a}dx$. The quantity C_n is expressed in terms of a well-defined integral so $V(x,y)$ is determined.

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2. *Electric Dipoles*

(a) (7) Find the dipole moment of a straight wire of length L with a linear charge density of the form $\lambda(z) = \lambda_0 z/L$ for $|z| < L/2$.

In general $\mathbf{p} = \int d^3r\, \mathbf{r}\rho(\mathbf{r})$. In this problem the charge density is a linear (a) or surface charge (b).

The dipole moment is in the z direction with magnitude
$p = \int_{-L/2}^{L/2} \lambda(z) z dz = \int_{-L/2}^{L/2} \frac{\lambda_0 z}{L} z dz = \lambda_0 \frac{z^3}{3L}\Big|_{-L/2}^{L/2} = \lambda_0 \frac{L^2}{12}$.

(b) (6) Find the dipole moment of a hollow sphere of radius R with a surface charge distribution $\sigma(\theta) = (q/R^2)\cos\theta$.

The dipole moment is in the z direction with magnitude

$p = \int \sigma(\theta) z\, dA, \quad z = R\cos\theta, \quad dA \to 2\pi \sin\theta d\theta R^2$

So $p = 2\pi(q/R^2)R^2 \int_0^\pi \sin\theta d\theta\, \cos\theta\, R\cos\theta = 2\pi qR(2/3)$

(c) (7) Find the force on an electric dipole $\mathbf{p}$ located a separation $\mathbf{r}$ away from a point charge q.

The force $\mathbf{F}$ on a dipole $\mathbf{p}$ is given by $\mathbf{F} = \mathbf{p}\cdot\boldsymbol{\nabla}\,\mathbf{E} = \boldsymbol{\nabla}(\mathbf{p}\cdot\mathbf{E})$ because $\mathbf{p}$ is a constant vector. Here $\mathbf{E} = \frac{q}{4\pi\epsilon_0}\frac{\mathbf{r}}{r^3}$. So

$\mathbf{F} = \frac{q}{4\pi\epsilon_0}[\frac{\boldsymbol{\nabla}(\mathbf{p}\cdot\mathbf{r})}{r^3} + \mathbf{p}\cdot\mathbf{r}\boldsymbol{\nabla}\frac{1}{r^3}] = \frac{q}{4\pi\epsilon_0}[\frac{\mathbf{p}}{r^3} - 3\frac{\mathbf{p}\cdot\mathbf{r}}{r^4}]$

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3. *Circular line charge* A line charge in the shape of a circle of radius R is centered at the origin and lies in the xy plane. The linear charge density is given as λ.

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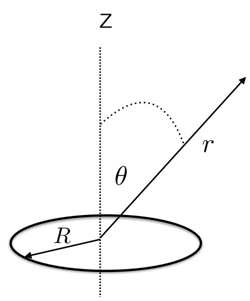

(a) (2) Determine Q, the total charge on the ring.

$Q = 2\pi\lambda R$

(b) (4) The electric potential can be written as $V(r,\theta) = \sum_{l=0}^{\infty} f_l(r) P_l(\cos\theta)$, where $f_l(r)$ is an unspecified function that depends on l and the distance from the origin r, at all points in space that are not on the line charge. Determine the values of l for which the function $f_l(r)$ must vanish. Explain.

There is a top down symmetry here. The geometry is invariant if θ is replaced by $\pi - \theta$. This is reflection symmetry about the xy plane. This means that $V(r,\theta) = V(r, \pi - \theta)$. $P_l(\theta) = P_l(\pi - \theta)$ if l is an even integer and $P_l(\theta) = -P_l(\pi - \theta)$ if l is odd. Therefore only even values of l enter.

(c) (4) Determine the electric potential $V(z)$ at a position along the z axis.

For observation points along the z axis, all points on the ring are the same distance $\sqrt{z^2 + R^2}$ away from the observation point. Thus

$V(z) = \frac{Q}{4\pi\epsilon_0} \frac{1}{\sqrt{z^2+R^2}}$.

(d) (10) Determine the electric potential for positions such that $r < R$. You *may* define the answer to part (c) as $V(z) = \frac{Q}{4\pi\epsilon_0} \sum_n C_n z^n$ for $z < R$, and take C_n as given. You may also taka $P_l(x)$ as given.

For $r < R$ the general solution of Laplace's equation with the azimuthal symmetry of this problem may be written as $V(r,\theta) = \frac{Q}{4\pi\epsilon_0} \sum_{l=0} r^l A_l P_l(\cos\theta)$, with A_l unknown. For points on the z axis this becomes $V(r, \theta = 0) = V(z) = \frac{Q}{4\pi\epsilon_0} \sum_{l=0} A_l r^l$ because $\cos 0 = 1$ and $P_l(1) = 1$. The answer to part (b) is $V(z) = \frac{Q}{4\pi\epsilon_0} \sum_n C_n z^n = \frac{Q}{4\pi\epsilon_0} \sum_l C_{2l} z^{2l}$ because $V(z)$ depends on z^2. For points on the z axis $z^2 = r^2$. Thus $A_l = C_{2l}$ and $V(r,\theta) = \frac{Q}{4\pi\epsilon_0} \sum_{l=0} r^l C_{2l} P_l(\cos\theta)$.

Alternate method (either method is ok) : In class we learned that $\frac{1}{|\mathbf{r}-\mathbf{r}'|} = \frac{1}{\sqrt{r^2+r'^2-2\mathbf{r}\cdot\mathbf{r}'}} = \sum_{l=0} \frac{r^l}{r'^{l+1}} P_l(\hat{\mathbf{r}} \cdot \hat{\mathbf{r}}')$ if $r < r'$. For points $\mathbf{r}$ on the z-axis $\hat{\mathbf{r}} \cdot \hat{\mathbf{r}}' = 0$ and $P_l(0) = 0$ if l is an odd integer. Thus $\frac{1}{\sqrt{z^2+R^2}} = \sum_{l=0\text{even}} \frac{r^l}{R^{l+1}} P_l(0)$ with $z^2 = R^2$. This expression must match the expression (above) for $V(r, \theta = 0)$ in terms of Legendre polynomials. Thus the coefficient $A_l = P_l(0)$.