PHYSICS 321:
ELECTROMAGNETISM

24 OCT. 2019 Midterm 1

Name: ______________________________ Stu. ID. ____________________

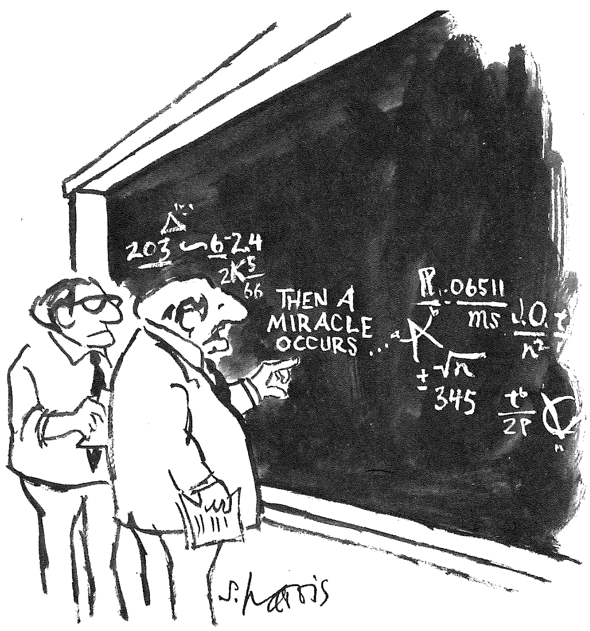
Section Number ______________________________

There are four problems in this exam, each with several parts. Each problem is worth 20 points. Please do not open the test until the starting time is announced.

Read the problems carefully.

The equations below could be useful.
$\int xe^{ax} = \frac{e^{ax}}{a^2}(ax-1)dx$
$\int x^2e^{ax} = \frac{e^{ax}}{a^3}(a^2x^2-2ax+2)dx$
$\int x^3e^{ax} = \frac{e^{ax}}{a^4}(a^3x^3-3a^2x^2+6ax-6)dx$

Read the problems carefully.



"I think you should be more explicit here in step two."

321 Midterm I solution

1. *Potential energy* Four point charges, each of charge q and mass m are held fixed and located at the corners of a square of side L. You may ignore gravitational potential energy throughout this problem.

(a) (10) Determine the potential energy of this configuration.

The potential energy $W = \frac{1}{8\pi\epsilon_0} \sum_{i,j\neq i} \left[\frac{q_i q_j}{|\mathbf{r}_i - \mathbf{r}_j|} \right]$.

Each of the four charges is a distance L away from two other charges and a distance $\sqrt{2}L$ away from a third charge. Therefore $W = \frac{4}{8\pi\epsilon_0} \left[\frac{2q^2}{L} + \frac{q^2}{\sqrt{2}L} \right] = \frac{q^2}{2\pi\epsilon_0 L}(2 + \frac{1}{\sqrt{2}})$

(b) (10) The charges in this configuration are released from rest. Use *conservation* of energy to determine the speed of one of the charges a very long time after the release. Define your answer to part (a) as W and express your answer in terms of W.

The kinetic energy of the four charges (each of mass m) at a very long time after their release (so that $L \to \infty$ and the final electric potential energy is 0) is given by conservation of energy to be equal to the initial electrical potential energy W so that

$4(\frac{1}{2}mv^2) = W$ so that $v = \sqrt{\frac{W}{2m}}$

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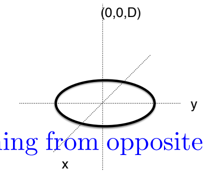
2. *Ring of charge*

This problem concerns *ring* of charge of radius R in the xy plane and centered at the origin. The linear charge density is given as λ



(a) (5) Determine the x and y components of $\mathbf{E}$ for a position a distance D above the origin. Explain your answer. Explain your answer.

At positions along the axis the contributions to the horizontal (x and y) components of $\mathbf{E}$ coming from opposite points of the ring cancel. Thus $E_x = 0$ and $E_y = 0$

(b) (5) Evaluate the electric field $\mathbf{E}$ for a position a distance D above the origin:

$\mathbf{E}(\mathbf{r}) = \frac{1}{4\pi\epsilon_0}\int dl'\lambda\frac{(\mathbf{r}-\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|^3}$ For the line integral $dl' = Rd\phi'$, with ϕ' running from 0 to 2π. The source point $\mathbf{r}' = (R\cos\phi', R\sin\phi', 0)$ the observation point $\mathbf{r} = (0,0,z)$. The distance $|\mathbf{r}-\mathbf{r}'| = \sqrt{z^2+R^2}$. The horizontal components of $\mathbf{E}$ vanish so that $\mathbf{E} = \hat{\mathbf{k}}\frac{\lambda R}{4\pi\epsilon_0}\int_0^{2\pi} d\phi' \frac{z}{(z^2+R^2)^{3/2}} = \hat{\mathbf{k}}\frac{2\pi\lambda R}{4\pi\epsilon_0}\frac{z}{(z^2+R^2)^{3/2}}$

(c) (5) Consider the electric potential $V(x,y,z)$ for positions that are not on the ring. Determine $\nabla^2 V$. Explain your answer.

Poission's equation says that $\nabla^2 V = -\frac{\rho}{\epsilon_0}$. But $\rho = 0$ for positions that are not on the ring. Thus $\nabla^2 V = 0$

(d) (5)Write an expression for $V(x,y,z)$ in terms of a one-dimensional integral. **Don't evaluate the integral.**
$V(x,y,z) = \frac{1}{4\pi\epsilon_0}\int dl'\lambda\frac{1}{|\mathbf{r}-\mathbf{r}'|}$.

Now the distance $|\mathbf{r}-\mathbf{r}'| = \sqrt{z^2+(x-R\cos\phi')^2+(y-R\sin\phi')^2} = \sqrt{r^2+R^2-2xR\cos\phi'-2yR\sin\phi'}$, where $r^2 = x^2+y^2+z^2$.

Thus $V(x,y,z) = \frac{\lambda R}{4\pi\epsilon_0}\int_0^{2\pi}\frac{1}{\sqrt{r^2+R^2-2xR\cos\phi'-2yR\sin\phi'}}\,d\phi'$.

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3. A region of space contains a spherically symmetric charge density $\rho(r)$, where r is the distance from a specified origin. In particular, take $\rho(r) = q\delta(\mathbf{r}) - q\frac{\mu^2 e^{-\mu r}}{4\pi r}$, where $\delta(\mathbf{r})$ is the three-dimensional Dirac delta function. This is the charge density of an atom with a positive point charge at the center surrounded by a negative charge distribution smeared out over space.

(a) (8) Consider an imaginary sphere (Gaussian sphere) of radius $R > 0$, centered at the origin. Evaluate the amount of charge (define it as $Q(R)$) within the Gaussian sphere.

By definition of density $Q(R) = \int d^3r\rho(\mathbf{r}) = \int d^3r(q\delta(\mathbf{r}) - q\frac{\mu^2 e^{-\mu r}}{4\pi r}) = q - q\int d^3r\frac{\mu^2 e^{-\mu r}}{4\pi r} = q - q\int_0^R r^2\frac{\mu^2 e^{-\mu r}}{r}\,dr = q - q\mu^2\int_0^R re^{-\mu r}\,dr$.

The integral $\int xe^{ax}dx = \frac{e^{ax}}{a^2}(ax-1)$ is given on the first page of the exam. Here $a = -\mu$, $x = r$,

so $\int re^{-\mu x}dr = -\frac{e^{-\mu r}}{\mu^2}(\mu r+1)$. The definite integral $\int_0^R re^{-\mu x}dr = -\frac{e^{-\mu R}}{\mu^2}(\mu R+1) + 1/m^2$

Thus $Q(R) = q - q\mu^2(-\frac{e^{-\mu R}}{\mu^2}(\mu R+1) + 1/\mu^2) = qe^{-\mu R}(\mu R+1)$. $Q(R)$ vanishes as R approaches ∞ as it must.

(b) (7) Now determine the electric field $\vec{E}$ at positions on the Gaussian sphere. Express your answer in terms of $Q(R)$, which you may now assume as given. Be sure to specify the direction of $\vec{E}$.

Spherical symmetry guarantees that $\mathbf{E}(\mathbf{R}) = E(R)\hat{\mathbf{r}}$ where $\hat{\mathbf{r}}$ points radially outward from the origin. Gauss Law says that $\oint_{\mathcal{S}}\mathbf{E}\cdot\mathbf{dA} = \frac{Q_{\text{enclosed}}}{\epsilon_0}$. Here $\mathbf{dA} = \hat{\mathbf{r}}dA$ so $\int_{\mathcal{S}}\mathbf{E}\cdot\mathbf{dA} = E(R)\oint_{\mathcal{S}}dA = E(R)4\pi R^2 = \frac{Q(R)}{\epsilon_0}$.

This means that $\mathbf{E}(R) = \hat{\mathbf{r}}\frac{Q(R)}{4\pi\epsilon_0 R^2}$

(c) (5) Determine $\boldsymbol{\nabla}\cdot\mathbf{E}$ The first Maxwell equation says that $\boldsymbol{\nabla}\cdot\mathbf{E} = \frac{\rho}{\epsilon_0}$.

Thus in this problem $\boldsymbol{\nabla}\cdot\mathbf{E} = \frac{1}{\epsilon_0}(q\delta(\mathbf{r}) - q\frac{\mu^2 e^{-\mu r}}{4\pi r})$.