

9

why did this method work.

① Completeness

② Orthogonality

$$f(y) = \sum_{n=1}^{\infty} C_n f_n(y).$$

↑
arbitrary
function

↑
Complete Set.

$$\int_a^b f_n(y) f_m(y) dy = 0 \quad n \neq m$$

↑
orthogonal.

Important General Principle

All solutions of

$$-\frac{d}{dy} \left(\tau(y) \frac{df(y)}{dy} \right) + V(y) f(y) = \omega^2 \sigma(y) f(y)$$

for $\tau \neq 0$ or $\omega \neq 0$ in where $a \leq y \leq b$

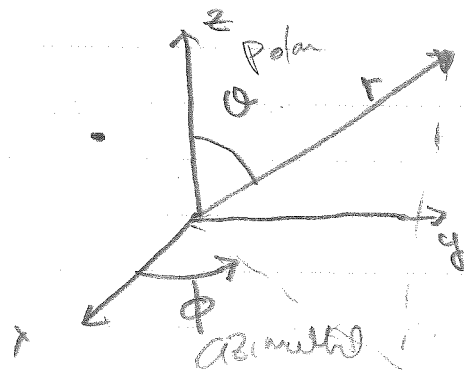
Satisfies with $\tau(y) > 0$ $\sigma(y) > 0$ $a \leq y \leq b$
with average of BC as
complete orthogonal Sturm-Liouville eq

Laplace's eqn "

Spherical - coord mater

$$\nabla^2 V = 0$$

$$\nabla^2 =$$



$$z = r \cos \theta$$

$$x = r \sin \theta \cos \phi$$

$$y = r \sin \theta \sin \phi$$

$$V(x, y, z) \rightarrow V(r, \theta, \phi)$$

$$\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$$

$$\nabla^2 = \frac{1}{r^2} \frac{\partial}{\partial r} r^2 \frac{\partial}{\partial r} + \frac{1}{r^2 \sin \theta} \frac{\partial}{\partial \theta} \sin \theta \frac{\partial}{\partial \theta} + \frac{1}{r^2 \sin^2 \theta} \frac{\partial^2}{\partial \phi^2}$$

Study $\frac{1}{r} \frac{\partial}{\partial r} r^2 \frac{\partial}{\partial r} \psi$

$$= \frac{1}{r} \left[2r \frac{\partial \psi}{\partial r} + r^2 \frac{\partial^2 \psi}{\partial r^2} \right]$$

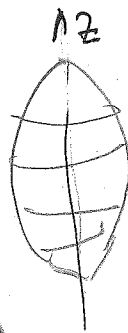
$$= \left(2 \frac{\partial}{\partial r} + \frac{\partial^2}{\partial r^2} \right) \psi$$

$$= \frac{1}{r} \frac{\partial^2}{\partial r^2} (r \psi)$$

3 D problems

We are going to solve Lap in 3 D

Special case



Azimuthal symmetry

$$V(\vec{r}) = V(r, \theta)$$

$$\nabla^2 V = 0$$

$$\frac{\partial V}{\partial \phi} = 0$$

From $r^2 \nabla^2 V$

$$\frac{\partial}{\partial r} r^2 \frac{\partial V}{\partial r} + \frac{1}{\sin \theta} \frac{\partial}{\partial \theta} \sin \theta \frac{\partial V}{\partial \theta} = 0$$

$$V(r, \theta) = R(r) \Theta(\theta)$$

$$\frac{1}{R} \frac{d}{dr} r^2 \frac{dR}{dr} + \frac{1}{\Theta \sin \theta} \frac{d}{d\theta} \sin \theta \frac{d\Theta}{d\theta} = 0$$

function of r

function of θ

$$\text{Let } \frac{1}{R} \frac{d}{dr} r^2 \frac{dR}{dr} = l(l+1)$$

Choice is made
so things come out
neat

$$\frac{1}{\Theta \sin \theta} \frac{d}{d\theta} \sin \theta \frac{d\Theta}{d\theta} = -l(l+1)$$

$$r^2 \frac{dR}{dr} = \lambda r^{2-1} r^2 = \lambda r^{2+1}$$

$$\text{so } \lambda(\lambda+1) = l(l+1)$$

clearly $\lambda = l$ is one sol.
 * Is that only soln?

No q.e.s

$$\lambda^2 - \lambda - l(l+1) = 0$$

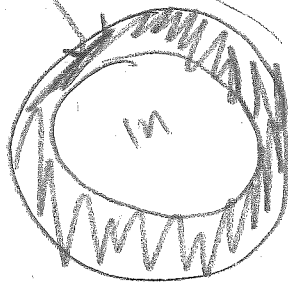
$$\lambda = \frac{-1 \pm \sqrt{1 + 4l(l+1)}}{2} = \frac{-1 \pm (2l+1)}{2}$$

$$\lambda = l, -(l+1)$$

Sand but it's

$$R(r) = A r^l + \frac{B}{r^{l+1}}$$

$$\begin{aligned} \lambda(\lambda+1) &= -l(l+1)[-l(l+1)+1] \\ &= -l(l+1)l \\ &= l(l+1) \end{aligned}$$



out

x class which is which

In general 2nd order D.E has 2 ind soln
 res, irregular at origin

Azimuthal Sym

$$V(r, \theta) = R(r) \Theta(\theta)$$

$$R(r) = Ar^l + \frac{B}{r^{l+1}}$$

$$\frac{1}{\sin \theta} \frac{d}{d\theta} \left(\sin \theta \frac{d\Theta}{d\theta} \right) = -l(l+1) \Theta$$

Special differential eq with special

solutions $\Theta(\theta) \Rightarrow P_l(\cos \theta)$ for $-1 \leq \cos \theta \leq 1$

Legendre polynomial of order $l = l = \text{non-negative integer}$

$$x = \cos \theta$$

$$\frac{d}{d\theta} = \frac{dx}{d\theta} \frac{d}{dx} = -\sin \theta \frac{d}{dx}$$

$$\frac{1}{\sin \theta} \frac{d}{d\theta} \left(\sin \theta \frac{d\Theta}{d\theta} \right) = \frac{1}{\sin \theta} \left(-\sin \theta \frac{d}{dx} \right) \left(-\sin \theta \frac{d\Theta}{dx} \right)$$

$$= \frac{d}{dx} \left((1-x^2) \frac{d\Theta}{dx} \right)$$

$$\frac{d}{dx} \left((1-x^2) \frac{d}{dx} P_l(x) \right) + l(l+1) P_l(x) = 0 \quad (A)$$

Sturm Liouville form

want solution to be finite, single valued

and continuous $-1 \leq x \leq 1$

$$x \rightarrow -x \quad \frac{d}{dx} \left((1-x^2) \frac{d}{dx} \right) \text{ is unchanged}$$

7. Solutions are polynomials

General properties

- or P_ℓ is a polynomial of order ℓ

$$P_\ell(x) = \sum_{j=0}^{\ell} a_j x^j \quad \ell = \text{integer}$$

$$\begin{aligned} \ell \text{ is either even or odd} \\ P_\ell(x) &= P_\ell(-x) \quad \text{if } \ell = \text{even} \\ &= -P_\ell(-x) \quad \text{if } \ell = \text{odd} \end{aligned}$$

$$\text{if } x \rightarrow -x$$

unchanged

Don't
forget!

$$\frac{d}{dx} (1-x^2) \frac{d}{dx} P_\ell(x) = -\ell(\ell+1) P_{\ell-1}(x) \quad (B)$$

$P_\ell(x), P_\ell(-x)$ obey same d.e.

$$P_\ell(x) = C_\ell P_\ell(-x) \quad \text{if } x \rightarrow -x \quad (A)$$

$$P_\ell(-x) = C_\ell P_\ell(x) \quad \text{if } x \rightarrow -x \quad (B)$$

$$1 \cdot C_\ell = -\ell(\ell+1) C_\ell P_\ell(x)$$

$$P_\ell(-x) = C_\ell P_\ell(x) = C_\ell^2 P_\ell(-x)$$

$$C_\ell^2 = 1$$

$$C_\ell = \pm 1$$

all ℓ !!

$$P_0(x) = 1$$

$$P_1(x) = x$$

$$P_2(x) = \frac{3x^2-1}{2}$$

$$P_3(x) = \frac{5x^3-3x}{2}$$

$$P_\ell(x=1) = 1$$

normalization

For class

$$x^2 \neq \sum a_n P_n(x)$$

which $a_n \neq 0$

SKIP

Orthogonality Just say result

$$P_n \left(\frac{d}{dx} (1-x^2) \frac{d}{dx} P_l \right) = -l(l+1) P_l P_n$$

$$P_l \left(\frac{d}{dx} (1-x^2) \frac{d}{dx} P_n \right) = -l(l+1) P_n P_l \quad n \neq 0$$

Integrate both sides

$$\int_{-1}^1 dx \left(P_n \frac{d}{dx} (1-x^2) \frac{d}{dx} P_l \right) = \int_{-1}^1 dx P_l \frac{d}{dx} (1-x^2) \frac{d}{dx} P_n$$

$$= (-l(l+1) + n(n+1)) \int_{-1}^1 dx P_l P_n$$

$$\text{LHS} = \left(P_n \frac{d}{dx} (1-x^2) \frac{d}{dx} P_l - P_l \frac{d}{dx} (1-x^2) \frac{d}{dx} P_n \right) \Big|_{-1}^1 = 0$$

because $1-x^2=0$ for $x=\pm 1$

$$\Rightarrow \int_{-1}^1 dx \left(\frac{dP_n}{dx} (1-x^2) \frac{dP_l}{dx} \right) + \int_{-1}^1 dx \frac{dP_l}{dx} (1-x^2) \frac{dP_n}{dx}$$

$$\Rightarrow (n(n+1) - l(l+1)) \int_{-1}^1 P_l P_n = 0 \quad \text{if } l \neq n \quad \int_{-1}^1 P_l P_n = 0$$

$$\text{if } l=n \quad \int_{-1}^1 P_l P_l = \neq 0$$

orthogonal

$$\int_{-1}^1 dx P_l(x) P_{l'}(x) = \begin{cases} 0 & l \neq l' \\ \frac{2}{2l+1} & l = l' \end{cases} \quad P_l(x=1)=1$$

$$= \delta_{ll'} \frac{2}{2l+1}$$

$$\delta_{ll'} = \begin{cases} 0 & l \neq l' \\ 1 & l = l' \end{cases}$$

P_l are complete set of orthonormal

$$f(x) = \sum_{l=0}^{\infty} C_l P_l(x) \quad -1 \leq x \leq 1$$

Mult by $P_n(x)$ and integrate

$$\int_{-1}^1 dx P_n(x) f(x) = \sum_{l=0}^{\infty} C_l \int_{-1}^1 P_n(x) P_l(x) dx$$

$$= C_n \frac{2}{2n+1}$$

$$C_n = \left(\frac{2n+1}{2} \right) \int_{-1}^1 dx P_n(x) f(x)$$

For class

example

write

$$x^2 = \sum_{n=0}^{\infty} C_n P_n(x)$$

determine non-zero C_n which
ones non-zero $n=0, n=2$

which C_1

$$C_0 = \frac{1}{2} \int_{-1}^1 dx \cdot x^2 P_0 = \frac{1}{2} \int_{-1}^1 dx \cdot x^2 \cdot 1$$

$$C_2 = \frac{1}{2} \int_{-1}^1 dx \cdot x^2 \left(\frac{3x^2 - 1}{2} \right)$$

$$= \frac{1}{2} \left[\frac{3}{2} \frac{x^5}{5} - \frac{1}{2} \frac{x^3}{3} \right]_{-1}^1$$

$$= \frac{1}{2} \left(\frac{3}{10} - \frac{1}{6} \right) = \frac{3}{2} - \frac{5}{6} = \frac{2}{3}$$

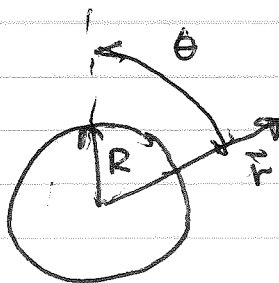
$$x^2 = \frac{1}{3} P_0 + \frac{2}{3} P_2(x)$$

$$\text{Check } x^2 = \frac{1}{3} + \frac{2}{3} \left(\frac{3}{2} x^2 - \frac{1}{2} \right) = x^2$$

Back to EOM

Example

Hollow sphere



with potential on surface

$$V(R, \theta) = V_0(\theta) \quad \text{given}$$

General solution

$$V(r, \theta) = \sum_{l=0}^{\infty} \left(A_l r^l + \frac{B_l}{r^{l+1}} \right) P_l(\cos \theta)$$

Problem want $V(r, \theta)$ everywhere

both $r < R$ and $r > R$

Start with inside $r < R$

$$V(R, \theta) = V_0(\cos \theta) = \sum_{l=0}^{\infty} A_l R^l P_l(\cos \theta)$$

multiply by $\int_{-1}^1 d\cos \theta P_n(\cos \theta)$

$$\int_{-1}^1 d\cos \theta P_n(\cos \theta) V_0(\cos \theta) = \frac{2n+1}{2} A_n R^n$$

note $\int_0^\pi \sin \theta d\theta = 2$

$$\int_0^\pi \sin \theta d\theta \int_{-1}^1 d\cos \theta P_n(\cos \theta) V_0(\theta) = I_n \quad A_n = \frac{2}{2n+1} \frac{I_n}{R^n}$$

Given A_n

we know the potential
inside $r < R$

What about outside?

Outside $A_l = 0$

$$V(r, \theta) = \sum_{l=0}^{\infty} \frac{B_l}{r^{l+1}} P_l(\cos \theta)$$

$V(r, \theta)$ must be continuous at $r = R$. This is because $\vec{E} = -\vec{\nabla} V$

$$\text{Thus } \sum_{l=0}^{\infty} \frac{B_l}{R^{l+1}} P_l(\cos \theta) = \sum_{l=0}^{\infty} A_l R^l P_l(\cos \theta)$$

By orthogonality of P_l we must have

$$\underline{B_l = A_l R^{2l+1}}$$

Example $V_0(\theta) = V_0 \cos^2 \theta$
 $x = \cos \theta$

From previous

$$V(\theta) = V_0 \left(\frac{P_0(x)}{3} + \frac{2}{3} P_2(x) \right)$$

which $A_n, B_n \neq 0$

$$r < R \quad V(r, \theta) = A_0 P_0(\cos \theta) + \underline{A_2} R^2 P_2(\cos \theta)$$

$$A_0 = V_0/3 \quad A_2 R^2 = \frac{2}{3} V_0$$

$$V(r, \theta) = V_0 \left(\frac{1}{3} + \frac{2}{3} \frac{r^2}{R^2} P_2(\cos \theta) \right)$$

Outside $r > R$

$$V(r, \theta) = \frac{B_0}{r} P_0 + \frac{B_2}{r^3} P_2(\cos \theta)$$

V is continuous

$$\frac{B_0}{R} = \frac{V_0}{3} \quad , \quad \frac{B_2}{R^3} = \frac{2}{3} V_0$$

$$V(r, \theta) = V_0 \left(\frac{R}{3r} P_0(\cos \theta) + \frac{2}{3} \frac{R^3}{r^3} P_2(\cos \theta) \right)$$

V_0 is continuous

$\vec{E}$ continuous?

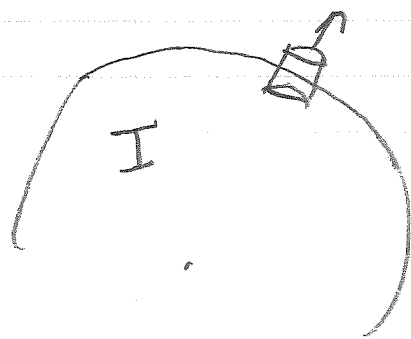
Surface charge density

II

Gauss Law

$$\frac{\sigma}{\epsilon_0} = (\mathbf{E}_{II} - \mathbf{E}_I) \cdot \hat{\mathbf{r}}$$

$$= \left(-\frac{\partial V_{II}}{\partial r} + \frac{\partial V_I}{\partial r} \right) \Big|_{r=R}$$



$$= V_0 \left[\frac{R}{3R^2} + \frac{2\beta}{3} \frac{R^3}{R^4} P_2(\cos\theta) \right]$$

$$+ V_0 \left[\frac{4}{3} \frac{1}{R} P_2(\cos\theta) \right]$$

$$= \frac{V_0}{R} \left[\frac{1}{3} + 2 P_2(\cos\theta) + \frac{4}{3} P_2(\cos\theta) \right]$$

$$= \frac{V_0}{3R} \left[1 + \frac{10}{3} P_2(\cos\theta) \right]$$

$$= \frac{V_0}{3R} \left(1 + \frac{10}{3} \frac{(\cos^2\theta - 1)}{2} \right)$$

$$1 - \frac{5}{3}$$

$$= \frac{V_0}{3R} (15 \cos^2\theta - 4)$$

Conducting sphere in uniform field

Before

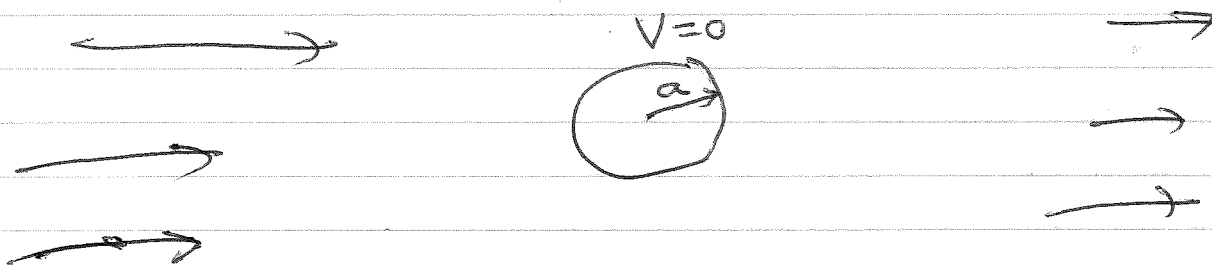


$$V(r, \theta) = -E_0 z = -E_0 r \cos \theta$$

$$= -E_0 r' P_1(\cos \theta)$$

After

After



V inside and outside sphere

at large distances must have result above

$V(r, \theta) = ?$ use general solution

$A_1 = -E_0$ all A_l ($l \neq 1$) = 0

Continuity $V(r=a, \theta) = 0$

$$0 = a A_1 \cos \theta + \frac{B_1}{a^2} \cos \theta$$

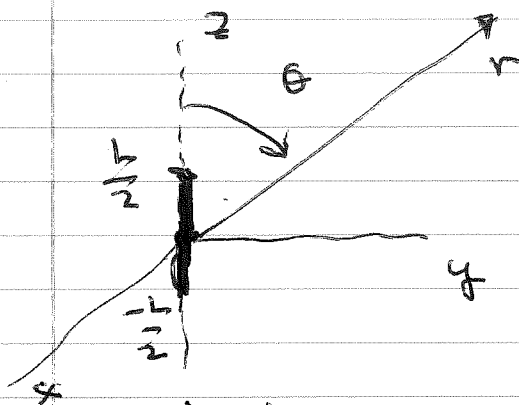
$$B_1 = -a^3 A_1 = E_0 a^3$$

$$V(r, \theta) = \left(-E_0 r + \frac{E_0 a^3}{r^2} \right) \cos \theta$$

what about B_l for $l \neq 1$?

Another application of azimuthal symmetry, Warning this is not in textbook, but very likely will be on exam(s)!!
 Will do this 3+ times

$V(r, \theta)$ for a line charge of finite length



$\lambda = \text{constant charge per unit length}$
 $Q = \lambda L$ total charge

Let's get $V(r, \theta)$ for $r > L/2$

$$V(r, \theta) = \sum_{l=0}^{\infty} \frac{B_l}{r^{l+1}} P_l(\cos \theta)$$

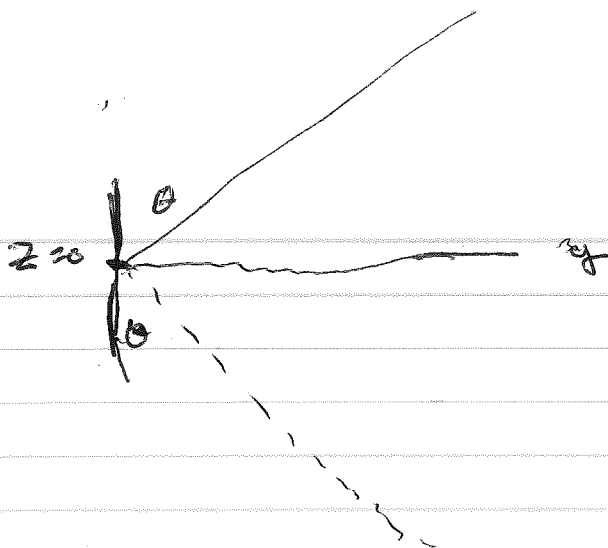
Laplace's eqn

First look to see which B_l are not zero

What's the symmetry of ~~the~~ this problem

There is a symmetry here - can't tell

the top from the bottom



$$x = r \cos \theta$$

$$V(\theta) = V(\pi - \theta) \quad ; \quad \cos(\pi - \theta) = -\cos \theta$$

l must be even

Now how to get B_L for l even

Evaluate $V(r, \theta)$ at $\theta = 0$, easy to

evaluate $V(r, \theta = 0)$ directly for $r > L/2$

$$V(r, \theta = 0) = \frac{\lambda}{4\pi\epsilon_0} \int_{-L/2}^{L/2} \frac{dz'}{r - z'} \quad r > L/2$$

$$= \frac{\lambda}{4\pi\epsilon_0} (-) \ln(r - z') \Big|_{-L/2}^{L/2}$$

$$= \frac{\lambda}{4\pi\epsilon_0} \ln \frac{r + L/2}{r - L/2} = \frac{\lambda}{4\pi\epsilon_0} \ln \left(\frac{1 + L/2r}{1 - L/2r} \right)$$

Make series expansion

$$\ln \left(\frac{1 + \frac{L}{2r}}{1 - \frac{L}{2r}} \right) = \sum_{l \text{ even}} \frac{2}{l+1} \left(\frac{L}{2r} \right)^{l+1} \frac{1}{r^{l+1}}$$

$$V(r, \theta = 0) = \frac{\lambda}{4\pi\epsilon_0} \sum_{l=\text{even}} \sum_{l+1} \left(\frac{L}{2r}\right)^{l+1}$$

But we have another expression for $V(r, \theta = 0)$

$$\text{Because } P_l(\cos 0 = 1) = 1$$

$$V(r, \theta = 0) = \sum_{l=0}^{\infty} \frac{B_l}{r^{l+1}}$$

$$B_l = \frac{\lambda}{4\pi\epsilon_0} \sum_{l+1} \left(\frac{L}{2}\right)^{l+1} \quad \text{if } l = \text{even}$$

$$= 0 \quad \text{if } l = \text{odd}$$

$$V(r, \theta) = \frac{\lambda}{4\pi\epsilon_0} \sum_{l=\text{even}} \sum_{(l+1)} \left(\frac{L}{2r}\right)^{l+1} P_l(\cos \theta)$$

$r > \frac{L}{2}$

Math comment: suppose

$$f(x) = \sum_{n=0}^{\infty} a_n x^n \quad \text{and} \quad f(x) = \sum_{n=0}^{\infty} b_n x^n$$

Then $a_n = b_n$ (all n)

proof

$$f(0) = a_0 = b_0$$

$$\left. \frac{df}{dx} \right|_{x=0} = a_1 = b_1 \quad \text{etc}$$

Can you get $V(r, \theta)$ $r < \frac{L}{2}$ for positions away from charge?

Can get $V(r, \theta)$ on the line charge?

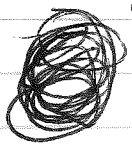
Large r

$$V(r, \theta) = \frac{\lambda L}{4\pi\epsilon_0} \left[\frac{1}{r} + \frac{1}{6} \frac{L^2}{r^3} P_2(\cos\theta) + \dots \right]$$
A diagram showing a rod of length L and a point at distance r and angle θ . An arrow points from the rod to the angle θ .

example of multipole expansion

General situation

have unknown something in a continuous distribution
either a charge density, mass

density, or density  ↑ localized

Physicists astronomers & geologists

make measurements outside the distribution,

Here let's say measure $V(r, \theta)$ outside distribution

What can we say about distribution $\rho(\vec{r})$?

$$\lim_{r \rightarrow \infty} V(r, \theta) = \frac{Q}{4\pi\epsilon_0 r} \quad \begin{array}{l} Q \text{ is total} \\ \text{charge} \end{array}$$

at $r \rightarrow \infty$ finite-sized charge distribution

looks like a point

$$\text{IF } V(r, \theta) = \sum_{l=0}^{\infty} \frac{B_l}{r^{l+1}} P_l(\cos\theta)$$

we might think that $\rho(\vec{r}) = \rho(r, \theta)$

no ϕ dependence

Want to say something more general

$\rho(\vec{r})$ localized, $\rho(\vec{r}) = \rho(r, \theta)$

azimuthal symmetry



$$V(\vec{r}) = \frac{1}{4\pi\epsilon_0} \int d^3r' \frac{\rho(\vec{r}')}{|\vec{r} - \vec{r}'|}$$

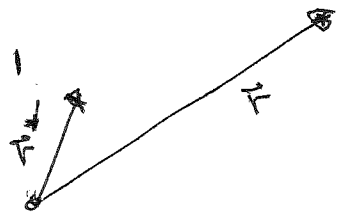
expand

in $P_L(\hat{r} \cdot \hat{r}')$ $\hat{r} \cdot \hat{r}' = \cos\theta$

If $\vec{r} \neq \vec{r}'$ Legendre's fired $\vec{r}'$

$$\frac{1}{|\vec{r} - \vec{r}'|} = \sum_n f_n(r, r') P_n(\cos\theta)$$

$q = 1$



let $r > r'$

L's eq. applies for r

$$\frac{1}{|\vec{r} - \vec{r}'|} = \sum_{n=0}^{\infty} \frac{B_n(r')}{r^{n+1}} P_n(\cos\theta)$$

let $\vec{r} \parallel \vec{r}'$

$$\hat{r} \cdot \hat{r}' = 1 \quad \cos\theta = 1$$



$$\begin{aligned} \frac{1}{|\vec{r} - \vec{r}'|} &= \frac{1}{r - r'} = \sum_{n=0}^{\infty} \frac{B_n(r')}{r^{n+1}} \cdot 1 \\ &= \frac{1}{r} \sum_{n=0}^{\infty} \left(\frac{r'}{r}\right)^n \end{aligned}$$

Thus

$r > r'$

$$\frac{1}{|\vec{r} - \vec{r}'|} = \sum_{n=0}^{\infty} \frac{r'^n}{r^{n+1}} P_n(\cos\theta)$$

if $r' > r$

$$\frac{1}{|\vec{r} - \vec{r}'|} = \sum_{n=0}^{\infty} \frac{r^n}{(r')^{n+1}} P_n(\cos\theta)$$

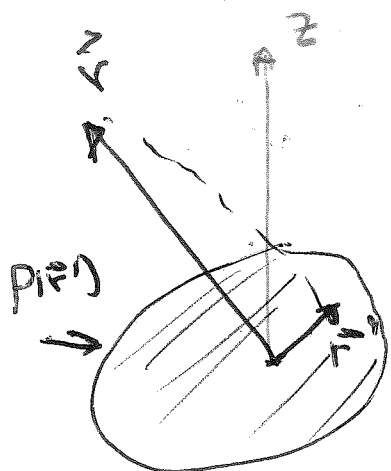
General

$$\frac{1}{|\vec{r} - \vec{r}'|} = \sum_{n=0}^{\infty} \frac{r_{>}^n}{r_{<}^{n+1}} P_n(\cos\theta)$$

$\vec{r} \neq \vec{r}'$

Now apply $\square$ to Potential
outside azimuthal symmetric charge
distribution

$$V(r, \theta) = \frac{1}{4\pi\epsilon_0} \int \frac{d^3r' \rho(r')}{|\vec{r} - \vec{r}'|}$$



$$= \frac{1}{4\pi\epsilon_0} \int d^3r' \sum_{l=0}^{\infty} \frac{r'^l}{r^{l+1}} P_l(\hat{r} \cdot \hat{r}') \rho(r')$$

IF $\rho(r') = \rho(r', \theta)$
azimuthal symmetry

We know

$$V(r, \theta) = \sum_{l=0}^{\infty} \frac{B_l}{r^{l+1}} P_l(\cos\theta)$$

now we can take $\vec{r}$ to be the z axis

Then $P_l(\hat{r} \cdot \hat{r}') = P_l(\cos\theta')$ and by

previous must have

$$B_l = \frac{1}{4\pi\epsilon_0} \int d^3r' r'^l P_l(\cos\theta') \rho(r', \theta')$$

B_l is a multipole coefficient