Last time

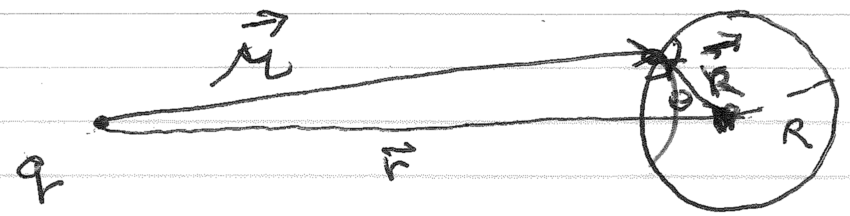
Poisson $\nabla^2 V = -\rho/\epsilon_0$ $\qquad \nabla^2 V = 0$ Laplace

Sol'ns of Laplace eq are average over nearby points in 1 & 2 dimension, no local maximum or minimum

Today 3 dimensions

In a region with $\rho = 0$ $V(\vec{r})$ is average of $V$ over a sphere centered at $\vec{r}$

Proof: Do for 1 point charge, then use superposition. Proof uses math techniques useful in remainder of academic year red arc line has constant $\theta$ where $\vec{\mathscr{r}}$ is the same




Claim $\quad V(\vec{r}) = \dfrac{1}{4\pi R^2} \oint da\, V(\vec{\mathscr{r}})$

$$\stackrel{?}{=} \frac{q}{4\pi\epsilon_0 r}$$

$$V(\vec{\mathscr{r}}) = \frac{q}{4\pi\epsilon_0} \frac{1}{|\vec{\mathscr{r}}|}$$

$$\vec{\mathscr{r}} = \vec{r} + \vec{R} \quad , \quad |\vec{\mathscr{r}}| = \sqrt{r^2 + R^2 + 2\vec{r}\cdot\vec{R}}$$

$$\vec{r}\cdot\vec{R} = -2rR\cos\theta \quad ; \quad da = R^2 \sin\theta\, d\theta\, d\phi$$
$$\rightarrow R^2 \sin\theta\, d\theta\, (2\pi)$$

need $\displaystyle\oint da\, V(\vec{\mathscr{r}}) = 2\pi R^2 \frac{q}{4\pi\epsilon_0} \int_0^{\pi} \sin\theta\, d\theta \frac{1}{\sqrt{R^2 + r^2 - 2rR\cos\theta}}$

Let $\quad x = \cos\theta$

$$\oint da\, V(\vec{R}) = 2\pi R^2 \frac{q}{4\pi\epsilon_0} \int_{-1}^{1} dx \frac{1}{\sqrt{R^2 + r^2 - 2rRx}}$$

$\oint da\, V(\vec{R}) =$

$$= \frac{2\pi R^2}{rR} (-) \sqrt{R^2 + r^2 - 2rRx}\,\Big|_{x=-1}^{1} \left(\frac{q}{4\pi\epsilon_0}\right)$$

$$= \frac{2\pi R}{r}\left[r+R - (r-R)\right] \left(\frac{q}{4\pi\epsilon_0}\right)$$

$$= \frac{4\pi R^2}{r} \cdot \frac{q}{4\pi\epsilon_0} = 4\pi R^2 \frac{q}{4\pi\epsilon_0 r}$$

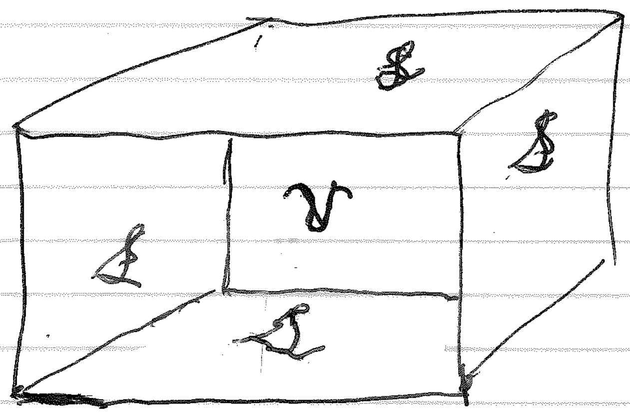
$$\frac{1}{4\pi R^2} \oint da\, V(\vec{R}) = \frac{q}{4\pi\epsilon_0 r} = V(r)$$

This means the solutions of Laplace's equation have no local max or min

# Uniqueness

Suppose you have a solution of Poisson's eqn that satisfies the boundary conditions. Claim: No other solutions exist.

In general some region of space with boundary conditions




More complicated pictures with holes in $\mathcal{V}$ are allowed

Typically $V$ is given on surface $\mathcal{S}$
or $\vec{\nabla} V$ " " " "
If $\mathcal{S}$ is conductor $\vec{\nabla} V = -\vec{E} =$
$= \sigma \hat{n}/\epsilon_0$

Uniqeness proof:

Suppose $V_1, V_2$ with both

$\nabla^2 V_1 = \rho/\epsilon_0$ , $\nabla^2 V_2 = \rho/\epsilon_0$

and $V_1 = V_2$ on $\underline{\mathcal{S}}$

or $\vec{\nabla} V_1 = \vec{\nabla} V_2$ on $\underline{\mathcal{S}}$

Claim $V_1 = V_2$ + constant

Proof $V_3 \equiv V_1 - V_2$

to show $V_3$ = constant (could be 0)

$\boxed{\text{what is } \nabla^2 V_3 \text{ ?}}$

$$\nabla^2 V_3 = \nabla^2 (V_1 - V_2) = -\frac{\rho}{\epsilon_0} + \rho/\epsilon_0 = 0$$

$\nabla^2 V_3 = 0$ $V_3 = 0$ on $\mathcal{S}$

or $\nabla V_3 = 0$ on $\mathcal{S}$

must get BC into game

$$\int_V d\tau\, V_3 \nabla^2 V_3 = 0$$

Convert to surface integral ?

Looks like integrate by parts

in 3 Dimension - get $\nabla V_3 \cdot \nabla V_3$ + Surface

Formal way use product rule for differentiation

$$\vec{\nabla} \cdot (V_3 \vec{\nabla} V_3) = \vec{\nabla} V_3 \cdot \nabla V_3 + V_3 \nabla^2 V_3$$

Thus

$$0 = \int_{v} d\tau \; V_3 \vec{\nabla}^2 V_3 = \int_{v} d\tau \left[ \vec{\nabla} \cdot (V_3 \vec{\nabla} V_3) - |\nabla V_3|^2 \right]$$

$$\int_{S} d\vec{a} \cdot (V_3 \nabla V_3) = 0$$ because $V_3$ or $\nabla V_3 = 0$

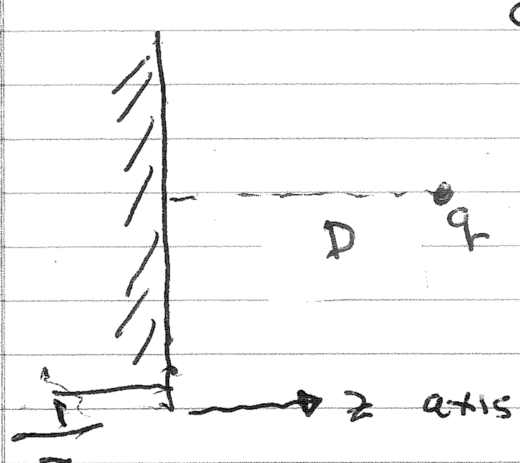
So must have $\qquad 0 = + \int |\nabla V_3|^2 \geq 0$

$$\vec{\nabla} V_3 = 0$$

So $V_3 = 0$ or a constant

# Boundary Value Problems - new method

Suppose have wall, with metal so a conductor attached to Earth which is also a decent conductor so at fixed potential $V=0$



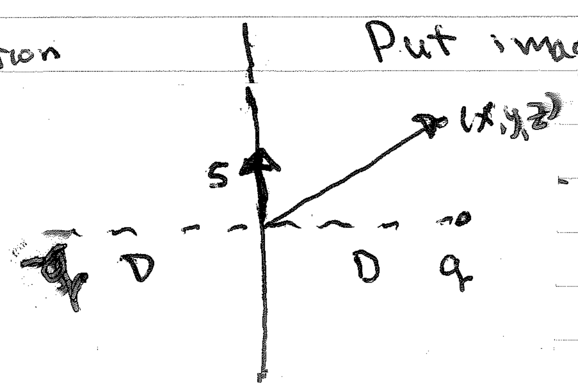

xy in wall. Take wall to be very large

Now bring in a + charge $q>0$ a distance D from the wall

What happens? conducting electrons run up the wall because of Coulomb attraction

So have Boundary Value problem. $V=0$ on wall $\nabla^2 V = -\frac{1}{\epsilon_0} q\,\delta(z-D)\,\delta(x)\,\delta(y)$

Solution




Put image charge a distance D behind wall

It's ok because we don't care about

$$V(x,y,z) = \frac{q}{4\pi\epsilon_0}\left[\frac{1}{\sqrt{(z-D)^2+x^2+y^2}} - \frac{1}{\sqrt{(z+D)^2+x^2+y^2}}\right]$$

symmetry $x^2+y^2 = s^2$

For a point on the wall $z=0$

$V(x,y,z=0) = 0$ so we have solved the problem. In the physical region $z \geq 0$, $V$ satisfies Poisson's eqn and the BC. $V$ is uniquely correct

Name: method of images

Given $V(x,y,z)$ what can we do with it?

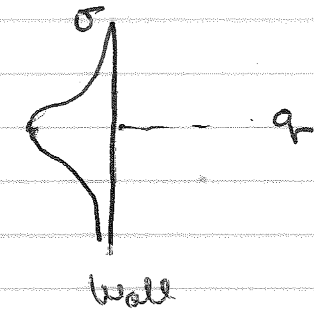
Where are the charges on the wall? $\vec{E} = \frac{\sigma}{\epsilon_0}\hat{n}$ where $\hat{n}$ is out of wall in $\hat{z}$ direction

$$\frac{\sigma}{\epsilon_0} = -\hat{n}\cdot\vec{\nabla}V$$

$$= -\frac{\partial V}{\partial z} \quad \text{at } z=0$$

$$-\frac{\partial V}{\partial z}\Bigg|_{z=0} = \frac{q}{4\pi\epsilon_0}\left[\frac{-(z-D)}{((z-D)^2+s^2)^{3/2}} - \frac{(z+D)}{((z+D)^2+s^2)^{3/2}}\right]\Bigg|_{z=0}$$

$$= -\frac{q}{4\pi\epsilon_0}\,\frac{2D}{(D^2+s^2)^{3/2}} = \frac{\sigma}{\epsilon_0}.$$




$$\sigma(s) = -\frac{q}{4\pi}\,\frac{2D}{(D^2+s^2)^{3/2}}$$

Reality check

Does this make sense : Largest at $s=0$

$D=0 \quad \sigma=0$ ; $D\to\infty$ , $\sigma\to 0$

What is total charge on wall?

Physics is not ...

$$q_{wall} = \int_{wall} da\, \sigma(s) = \int \frac{2\pi s\, ds}{4\pi}\left(-\frac{q\,2D}{(D^2+s^2)^{3/2}}\right)$$

$$= -\frac{q\,4\pi D}{4\pi}\int_0^\infty \frac{s\,ds}{(D^2+s^2)^{3/2}}$$

$$= (-qD)*(-)\,\frac{1}{(D^2+s^2)^{1/2}}\Bigg|_0^\infty = -q \;!$$

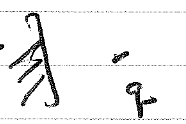
This is expected from Gauss Law

## More on boundary value problems

Why? many many apps in all areas of physics & engineering - building devices, detectors, power sources

So far we've see method of images

But this method is limited to special situations

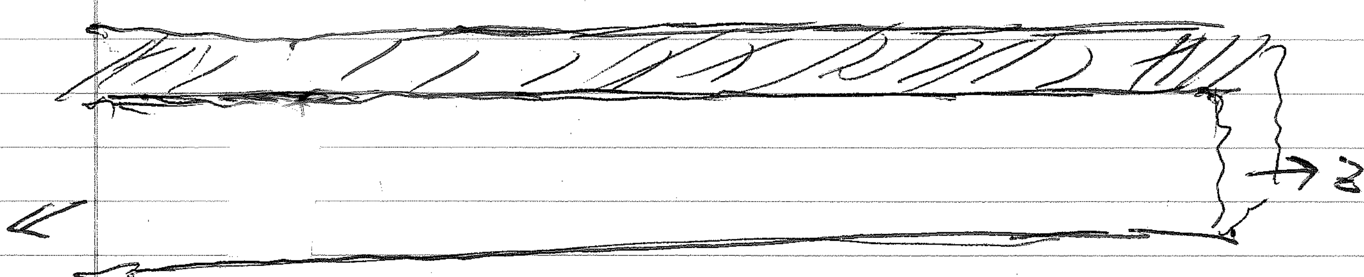
we've done 

And other examples are in the book

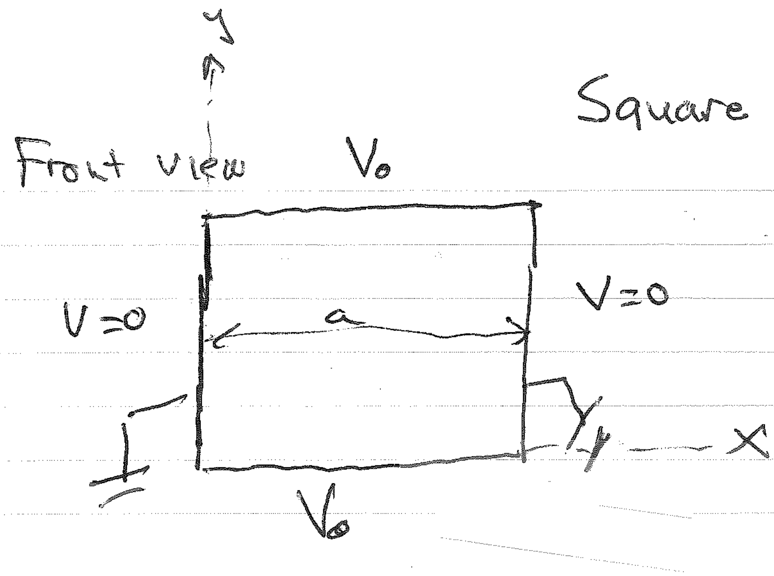
Start with two dimensional problem

Physics independent of position in one dimension

Example Very long square conducting pipe



Square of side length $= a$




$V(x,y) = 0$ @ $x = 0, a$

$V(x, y=0) = 0$
$V(x, y=a) = 0$

but $V = V_0$ at $y = \pm\, a/2$

Want sol'n Laplace eqn that satisfies B cond. Technique is funny

try $V(x,y) = X(x)\, Y(y)$ (1)

why so specialized?

$\infty$ many solutions - add em up to get Laplace eqn & BC

The PDE $\dfrac{\partial^2 V}{\partial x^2} + \dfrac{\partial^2 V}{\partial y^2} = 0$

becomes an ordinary D.E

$$Y(y)\frac{d^2X(x)}{dx^2} + X(x)\frac{d^2Y}{dy^2} = 0 \quad \text{Divide by } X(x)\,Y(y)$$

$$\underbrace{\frac{1}{X(x)}\frac{d^2X(x)}{dx^2}}_{\text{function of } x} = \underbrace{-\frac{1}{Y(y)}\frac{d^2Y}{dy^2}}_{\text{function of } y}$$

$\Rightarrow$ each must be a constant.

$$\frac{1}{X}X'' = c_1 \; ; \; \frac{1}{Y}Y'' = -c_1$$

The BC $X(0) = X(a) = 0$

The function $X(x)$ must be periodic sin or cos

Let $c_1 = -k^2$, $k$ is real

$X$ must vanish at ends $x = 0, a$

$$X = \sin kx \, , \quad k = \frac{n\pi}{a} \equiv k_n \quad n \geq 1$$

$$X = \sin\frac{n\pi x}{a}$$

Now $Y \to Y_n$ with $\frac{Y_n''}{Y_n} = k_n^2$

$Y_n$ is lin combo of cosh & sinh but

$Y_n$ is symmetric about $y = a/2$

$Y_n(y) = \cosh k_n(y - a/2)$, then $Y(a) = Y(0) = \cosh\frac{k_n a}{2}$

$$V(x,y) = \sum_{n=1} A_n \sin\frac{n\pi x}{a} \cosh\frac{n\pi}{a}(y-a/2)$$

$\cosh z$ is even

Must pick $A_n$ so that $\cosh z = \cosh(-z)$

$$V(x, y=a) = V(x, y=0) = V_0$$

So $$V_0 = \sum_{n=1} A_n \sin\frac{m\pi x}{a} \cosh\frac{n\pi}{2}$$

This is a Fourier series. Go thru math once

Multiply by $\sin\frac{m\pi x}{a}$ and integrate on $x$ from $0$ to $a$

RHS has $$\int_0^a dx \sin\frac{n\pi x}{a} \sin\frac{m\pi x}{a} = \delta_{nm}\frac{a}{2}$$

Thus $$A_m = \frac{2}{a} V_0 \int_0^a \sin\frac{m\pi x}{a} * \left(\frac{1}{\cosh\frac{m\pi}{2}}\right)$$

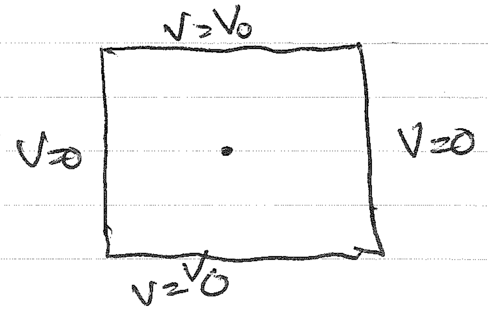
$$= \frac{2V_0}{a\cosh\left(\frac{m\pi}{2}\right)} \frac{a}{m\pi} (-) \cos\frac{m\pi x}{a}\Big|_0^a$$

$$A_m = \frac{4V_0}{\cosh\frac{m\pi}{2}} \frac{1}{m\pi} \text{ if } m \text{ is odd}$$

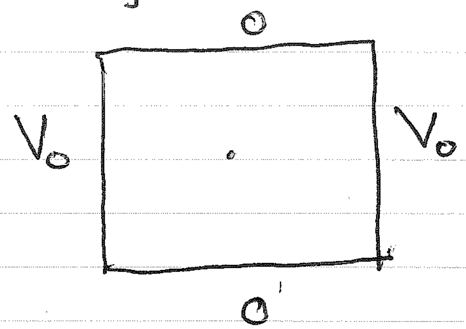
$$A_m = 0 \text{ if } m \text{ is even}$$

$$\boxed{V(x,y) = \sum_{m=\text{odd}} \frac{4V_0}{\cosh\frac{m\pi}{2}} \frac{1}{m\pi} \sin\frac{m\pi x}{a} \cosh\frac{m\pi}{a}\left(y-\frac{a}{2}\right)}$$

Solved this Problem

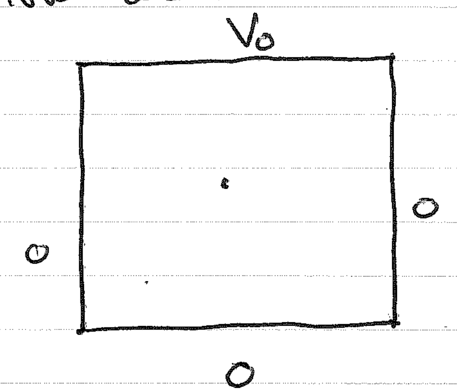



can you solve this ?



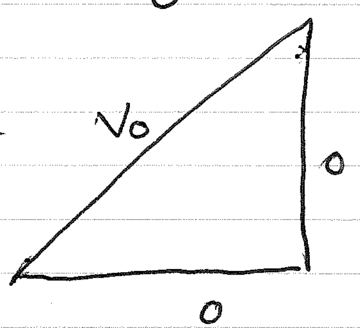

$(x \to y) \to (Y, X)$

How about ?




use $\sinh ky$ for $Y(y)$

How about




Rotate

no need to make $V_0$ a constant can have $V_0(x, y)$