Last time - energy of electric field and then properties of conductor.

Today we'll start by combining the two ideas: conductors used in energy storage and also in circuits. Under a capacitor is a terminal electrical component that stores energy in E-field.

Any two conductors separated by an insulator (or vacuum) form a capacitor:

\[ \begin{array}{cc}
\text{O} & \text{O} \\
Q & -Q
\end{array} \]

In practical applications, each conductor initially has 0 charge and electrons transform from one to the other.

Then two conductors have equal and opposite charge and total charge on the entire system remains 0.
Cap. with charge \( Q \) means one has to charge \(-Q\).

Cap. is used as components of electric circuits to block DC but allow AC.

Capacitors are used in resonant circuits to stabilize power flow. In circuits they are written as \( \frac{Q}{V} \) or \( \frac{\text{C}}{V} \) for conductors.

One way to charge a capacitor is to connect two wires to opposite terminals of a battery. Once a steady equilibrium is established, the battery is disconnected, this gives the potential difference between conductors:

\[
V_+ - V_- = - \int \frac{dQ}{e^2} = V
\]

\( V \) is proportional to \( Q \), more charge means more potential drop.

\[
C = \frac{Q}{V} = \frac{\text{Coulomb}}{\text{Volt}} = \text{Farad}
\]

Units [C] = \( \frac{\text{Coulomb}}{\text{Volt}} = \text{Farad} \)

\( \mu F = 10^{-6} F \)

\( PF = 10^{-12} F \)
The $C$ is a geom quantity
determined by sites shape, separation
of conductors
Worthwhlle to do example
Two concentric spherical metal
shells radii $a$, $b$

\[ \text{what is } C \]

Step 1 - what is $E$

Gauss Law, Gaussian surface
Spherical symmetry
\[ \Phi \cdot a^2 r < b \]

\[ E = \frac{Q}{4\pi \varepsilon_0 \ r^2} \]

\[ V = - \int_a^b E \cdot dl = + \int_b^a E \cdot dl \text{ path independent} \]

Take straight line, $dl = dr \ r$

\[ V = + \frac{Q}{4\pi \varepsilon_0} \int_a^b \frac{dr}{r^2} = - \frac{Q}{4\pi \varepsilon_0} \left[ \frac{1}{r} \right]_a^b \]

\[ = \frac{Q}{4\pi \varepsilon_0} \left( \frac{1}{b} - \frac{1}{a} \right) = \frac{Q}{4\pi \varepsilon_0} \left( \frac{b-a}{ab} \right) \]
\[ C = \frac{Q}{V} = \frac{4\pi \varepsilon_0 ab}{b-a} \]

How much energy is stored in a capacitor? To charge up a capacitor, remove electrons from one side and move them to the other. Do this means you fight against \( E \) which pulls electron back:

\[ q \]

How much work to get to final charge \( q \)? Suppose at intermediate stage, charge on positive plate is \( q \), \( V = q/C \)

The work to transport next piece of charge \( \Delta q \), is \( V \Delta q \)

\[ dW = \frac{q}{C} dq \]

\[ W = \frac{1}{2} \frac{q^2}{C} = \frac{1}{2} CV^2 \]
Next topic. Compute potentials

Two basic equations:

Poisson's: \( \nabla^2 V = -\frac{\rho}{\varepsilon} \)

If \( \rho = 0 \): Laplace \( \nabla^2 V = 0 \) (Harmonic function)

\( \nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} - \frac{\partial^2}{\partial z^2} \)

\( \nabla^2 \) = Laplacian

Now Laplace equation has one very easy solution - \( V = 0 \)

How can \( V \neq 0 \) when \( \nabla^2 V = 0 \)

\( \rho \neq 0 \) over here

Laplace equation in gravitation

Imagism, heat conduction, even quantum mechanics.

A version that's similar with similar solution techniques is

Helmholtz equation: \( (\nabla^2 + k^2) V = 0 \)
We'll follow & discuss solutions of L. eq in 1, 2, 3 dimensions.

1. Dimension - Physics independent of 2 dimensions - example very two large conductors.

\[ \nabla^2 V = 0 \Rightarrow \frac{d^2 V}{dx^2} = 0 \]

\[ V = ax + b \]

2nd order diff eq. 2 constants needed to specify potential.

Physics is nd - at \( x = 2 \text{ m} \), \( V(x) = 5 \text{ V} \)

\[ x = 4 \text{ m} \quad V(x) = 9 \text{ V} \]

What is \( a, b \)

\[ V(4) - V(2) = a(4 - 2) = 2a \text{ V} \]

\[ a = 2 \text{ V/m} \]

\[ V(2m) = 5 \text{ V} = \frac{2V(2) + b}{m} \]

\[ b = 1 \text{ V} \]
2 apparent features of 1D in case are general

1) \[ V(x) \text{ is average of } V(x+c) \theta V(x-c) \]
   \[ V(x) = \frac{1}{2} (V(x+c) + V(x-c)) \]
   \[ = \frac{1}{2} [a(x+c) + b + a(x-c) + b] \rightarrow 0 \]
   \[ = \frac{1}{2} (2ax + 2b) = ax + b \]

2) Solution of Laplace eq has no local max or min. Max or min at endpoint
   \[ \text{Local max or min looks like } L \lor U \]
   \[ \text{These are not solutions} \]
   \[ \text{Solution look like } L \]

More generally reasoning if \( V(x) \) were at a max it would be higher than at \( x \pm c \) and could not be the average
Two dimensions

\[ \frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} = 0 \]

This is a PDE.

\(
\text{infinitely many solutions:} \\
V = (mx + b)(ny + c) \\
or \quad e^{-xy} \sin x \\
\quad e^{-px} \sin by
\)

Can't be discussed in a simple form as in the 1 dim case

Same two properties true

\[ V(\mathbf{x}, \mathbf{y}) = \frac{1}{2\pi R} \int \limits_{\text{circle}} V \, d\mathbf{a} \]

\[ V \text{ is an average over circle} \]

This is an average over a circle
Example suppose

\[ V = (m x - 6)(m y + b) \]

\[ x = R \cos \phi \]
\[ y = R \sin \phi \]

Call point in question the origin

\[ \int_0^{2\pi} \int_0^{2\pi} r^2 R d\phi \left( m R \cos \phi + b \right) \left( m R \sin \phi + b \right) \]

\[ = \frac{1}{R} \int_0^{2\pi} d\phi \ a b = ab = V(0,0) \]

This average suggests a method for computer simulation solutions of Laplace's equation.

Start with \( V \) on boundary, guess \( V \) for grid of interior points first.

Pass \( V \) to each point by averaging of its neighbors, 12 nd passes repeats.

Method of relaxation is name
Again V has no local max or min. If it did V could not be the average of nearby points.

2 Dimensions

\[ \frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2} = 0 \]

Example \( V = (mx + c)(ny + e)z(1 - ez) \)

\[ \sin \alpha \sin \beta \ e^{\pm \sqrt{a^2 + b^2} z} \]

Again the property that \( V \) is average of nearby point holds.