Tuesday Oct. 15 2019

Energy of a distribution of charge

$W = \frac{1}{2}\sum_i q_i V(\vec{r}_i)$ $\qquad\qquad \sum_i q_i \to \int \rho\, d^3r$

$$W = \frac{1}{2}\int \rho(\vec{r})V(\vec{r})\,d^3r \quad \leftarrow$$ $\rho(\vec{r})$, $V(\vec{r})$ are evaluated at the same point in space

It's an interesting exercise to write W in terms of $\vec{E}$. It also gives some practice in vector calculus.

$$\rho(\vec{r}) = \epsilon_0 \vec{\nabla}\cdot\vec{E}(\vec{r})$$

$$W = \frac{1}{2}\epsilon_0 \int V(\vec{r})\left(\vec{\nabla}\cdot\vec{E}(\vec{r})\right)d^3r$$

Vector calculus version of integrating by parts.

<u>Reminder</u>: Integration by parts in 1D calculus. For 2 functions $u(x)$, $v(x)$

$$d(uv) = v\,du + u\,dv \qquad \text{(product rule)}$$

Integrate

$$\underbrace{\int d(uv)}_{=uv} = \int v\,du + \int u\,dv$$

so $$\int_a^b u\,dv = uv\Big|_a^b - \int_a^b v\,du.$$

Sometimes the rhs is easier to evaluate than the lhs.

One of the vector calculus versions:
$V(\vec{r})$ is a scalar field (number at every point $\vec{r}$)
$\vec{E}(\vec{r})$ is a vector field (vector at every point $\vec{r}$)

$V(\vec{r})\vec{E}(\vec{r})$ is a scalar times a vector
(which is just another vector) at every point $\vec{r}$.

So we can take a divergence and apply the product rule as in the 1D case

$$\vec{\nabla}\cdot(V\vec{E}) = \frac{\partial}{\partial x}(VE_x) + \frac{\partial}{\partial y}(VE_y) + \frac{\partial}{\partial z}(VE_z)$$

$$= E_x\frac{\partial V}{\partial x} + V\frac{\partial E_x}{\partial x} + E_y\frac{\partial V}{\partial y} + V\frac{\partial E_y}{\partial y} + E_z\frac{\partial V}{\partial z} + V\frac{\partial E_z}{\partial y}$$

so

$$\vec{\nabla}\cdot(V\vec{E}) = \vec{E}\cdot(\vec{\nabla}V) + V(\vec{\nabla}\cdot\vec{E})$$

(even if we don't use Cartesian coordinates)

Integrate over some volume

$$\int\vec{\nabla}\cdot(V\vec{E})\,d^3r = \int\vec{E}\cdot(\vec{\nabla}V)\,d^3r + \int V(\vec{\nabla}\cdot\vec{E})\,d^3r$$

Use divergence thm. to do the lhs and reshuffle:

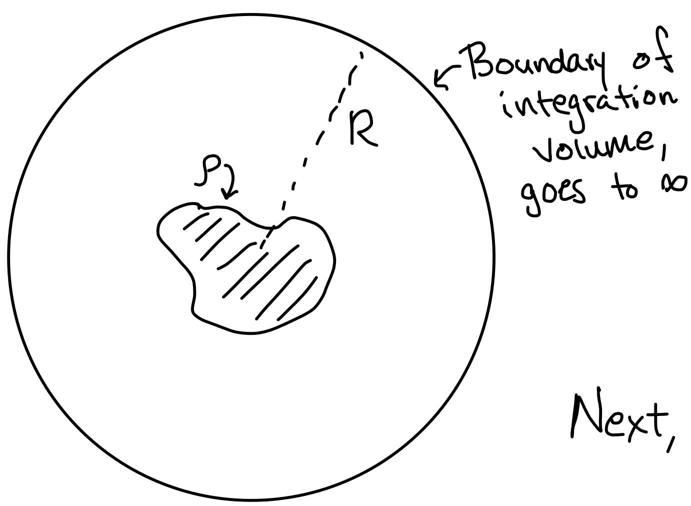
$$\int V(\vec{\nabla}\cdot\vec{E})\,d^3r = \oint_S (V\vec{E})\cdot d\vec{a} - \int\vec{E}\cdot(\vec{\nabla}V)\,d^3r.$$

Back to the physics

$$W = \frac{1}{2}\epsilon_0 \int V(\vec{r})\left(\vec{\nabla}\cdot\vec{E}(\vec{r})\right) d^3r$$

$$= \frac{\epsilon_0}{2}\oint_S V(\vec{r})\vec{E}(\vec{r})\cdot d\vec{a} \;-\; \frac{\epsilon_0}{2}\int \vec{E}(\vec{r})\cdot\left(\vec{\nabla}V(\vec{r})\right) d^3r$$

The integration volume needs to enclose all of $\rho(\vec{r})$. But if we make it bigger than $\rho(\vec{r})$, that's fine because $\rho=0$ out there so it doesn't change our answer. Let's take the volume to be a sphere of radius R and take $R\to\infty$. Far away, $E\sim\frac{1}{R^2}$, $V\sim\frac{1}{R}$, $da\sim R^2$ so the surface integral goes as $\frac{1}{R}\to 0$ as $R\to\infty$.



[We could leave R finite, but then we'd need to do the surface integral explicitly. We'd get the same answer in the end]

Next, we identify $-\vec{\nabla}V = \vec{E}$ so

$$W = \frac{\epsilon_0}{2}\int \vec{E}(\vec{r})\cdot\vec{E}(\vec{r})\, d^3r$$

$$\boxed{W = \frac{\epsilon_0}{2}\int \left|\vec{E}(\vec{r})\right|^2 d^3r}$$

This suggests the interpretation that $\frac{\epsilon_0}{2}\left|\vec{E}(\vec{r})\right|^2$ is the energy per unit volume stored in the electric field.

Anything wrong?

Our equation for W in terms of $\vec{E}$ is <u>manifestly positive</u>, whereas the formula in terms of ρ and V could be negative (e.g. two oppositely-charged point charges).

So which one is right?

Recall that the formula

$$W = \frac{1}{2}\sum_i q_i V(\vec{r}_i)$$

does not include the potential due to q_i in $V(\vec{r}_i)$, that is we sum up the potential due to all other charges, but omit the energy of q_i due to its own potential.

As an extreme example, consider a single point charge. Our original formula says $W=0$, since we omit the only source of V.

On the other hand, the conituous formula gives

$$W = \frac{1}{2}\int q\,\delta(\vec{r})\,\frac{q}{4\pi\epsilon_0 r}\,d^3r = \frac{q^2}{8\pi\epsilon_0}\cdot\frac{1}{0} = \infty.$$

or

$$W = \frac{\epsilon_0}{2}\int \left|\frac{q}{4\pi\epsilon_0 r^2}\right|^2 r^2\sin\theta\,dr\,d\theta\,d\phi = \frac{q^2}{8\pi\epsilon_0}\int_0^\infty \frac{1}{r^2}\,dr = \frac{q^2}{8\pi\epsilon_0}\left.\frac{-1}{r}\right|_0^\infty = \infty$$

The difference is that the discrete formula omits the (infinite) energy required to build a point charge. It only counts the energy required to arrange various pre-fabricated point charges.

<u>More detail</u>: If setting infinity to zero bothers you, you are not alone. Here's a second example that illustrates what's going on. Take two compact, localized charge distributions $\rho_1(\vec{r})$ and $\rho_2(\vec{r})$ so that $\rho(\vec{r}) = \rho_1(\vec{r}) + \rho_2(\vec{r})$.

We can compute the potentials $V_1(\vec{r})$ and $V_2(\vec{r})$ due to each distribution individually:

$$V(\vec{r}) = V_1(\vec{r}) + V_2(\vec{r}) = \frac{1}{4\pi\epsilon_0}\int \frac{\rho_1(\vec{r})}{r} d^3r + \frac{1}{4\pi\epsilon_0}\int \frac{\rho_2(\vec{r})}{r} d^3r$$

The total energy is

$$W = \frac{1}{2}\int \rho(\vec{r}) V(\vec{r})\, d^3r$$

$$= \frac{1}{2}\int \rho_1 V_1\, d^3r + \frac{1}{2}\int \rho_1 V_2\, d^3r + \frac{1}{2}\int \rho_2 V_1\, d^3r + \frac{1}{2}\int \rho_2 V_2\, d^3r$$

Now if we move ρ_1 as a whole to some other location, then the first and last terms don't change. Only the cross terms change.

So if ρ_1 and ρ_2 are "small" and we promise not to deform them, but only to translate them in space as a whole, then the first and last terms contribute a <u>finite</u> constant to W. Since (outside of general relativity) a constant added to the energy has no effect, we can ignore that bit without changing our physical predictions.

$$W = \frac{1}{2}\int \rho_1 V_2 \, d^3r + \frac{1}{2}\int \rho_2 V_1 \, d^3r + \text{const.}$$

In the limit that the distance between ρ_1 & ρ_2 is much bigger than the size of ρ_1 or ρ_2, then we can make the approximation

$$\rho_1 \approx q_1 \delta^3(\vec{r}-\vec{r}_1) \quad , \quad \rho_2 \approx q_2 \delta^3(\vec{r}-\vec{r}_2)$$

and

$$W = \frac{1}{2} q_1 V_2(\vec{r}_1) + \frac{1}{2} q_2 V_1(\vec{r}_2)$$

We <u>cannot</u> make the same approximation for the constant part. But if we're not worried about the constant, we can just ignore it, and using point charges makes the rest of the calculation easier.

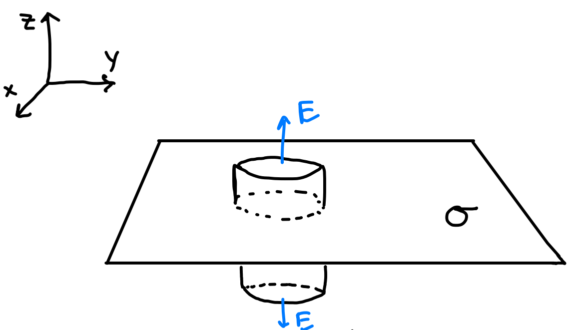
Boundary conditions for solving Poisson's equation

$$\nabla^2 V = -\rho/\epsilon_0$$

2nd order partial differential equation
$\rightarrow$ need 2 boundary conditions: V, $\vec{\nabla} V$

(Same story in mechanics: $F = m\frac{d^2}{dt^2}x \Rightarrow$ need $x(0)$, $v(0)$)

Example: infinite plane of charge



Above & Below
$\rho = 0 \Rightarrow \nabla^2 V = 0.$

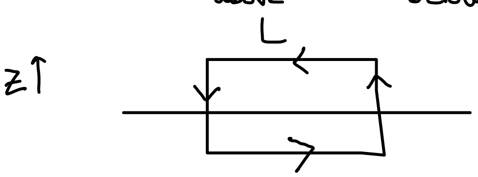
The boundary is at $z=0$.

What are the boundary conditions?

Gauss' law $\int \vec{E} \cdot d\vec{a} = \frac{Q_{encl}}{\epsilon_0} = \frac{\sigma A}{\epsilon_0}$

$$\vec{E}(z>0) = \frac{\sigma}{2\epsilon_0}\hat{k}, \quad \vec{E}(z<0) = -\frac{\sigma}{2\epsilon_0}\hat{k}$$

$$E^{\perp}_{above} - E^{\perp}_{below} = \frac{\sigma}{\epsilon_0}, \quad \vec{E}^{\parallel}_{above} - \vec{E}^{\parallel}_{below} = 0$$

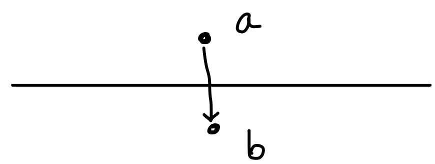


$$\int (\vec{\nabla} \times \vec{E}) \cdot d\vec{a} = \oint \vec{E} \cdot d\vec{l} = -L\left(\vec{E}^{\parallel}_{above} - \vec{E}^{\parallel}_{below}\right) = 0$$

This sets the BC for $\vec{E} = -\vec{\nabla} V$.

What about V ?

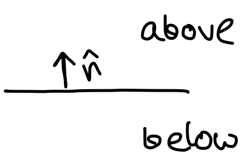
$$V_b - V_a = -\int_a^b \vec{E} \cdot d\vec{\ell}$$



As $a \to b$ $\int d\ell \to 0$ and $V_b - V_a \to 0$ or $V_a \to V_b$
V is continuous across the boundary.

These arguments apply in general, even if there is an additional external field $\vec{E}_{ext}$. If we get close enough to any surface, it eventually looks like an infinite plane. If there's no surface charge, just take $\sigma = 0$.

So, to summarize



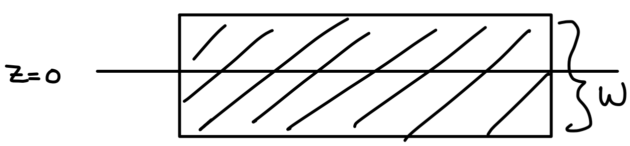
$$\boxed{\begin{aligned} &\vec{E}_{above} - \vec{E}_{below} = \frac{\sigma}{\epsilon_0}\hat{n} \\ &V_{above} - V_{below} \end{aligned}}$$

($\hat{n}$ points "above")

The potential and $\vec{E}$ field along the surface are continuous, and there's a discontinuity in the perpendicular or "normal" component of $\vec{E}$ which is proportional to the surface charge.

Another note about discontinuities and infinities. We get a discontinuity in $E^{\perp}$ because we have modeled the charge distribution as infinitessimally thin: $\rho(x,y,z) \approx \sigma(x,y)\,\delta(z)$. More realistically, we could spread the charge out over some finite thickness w

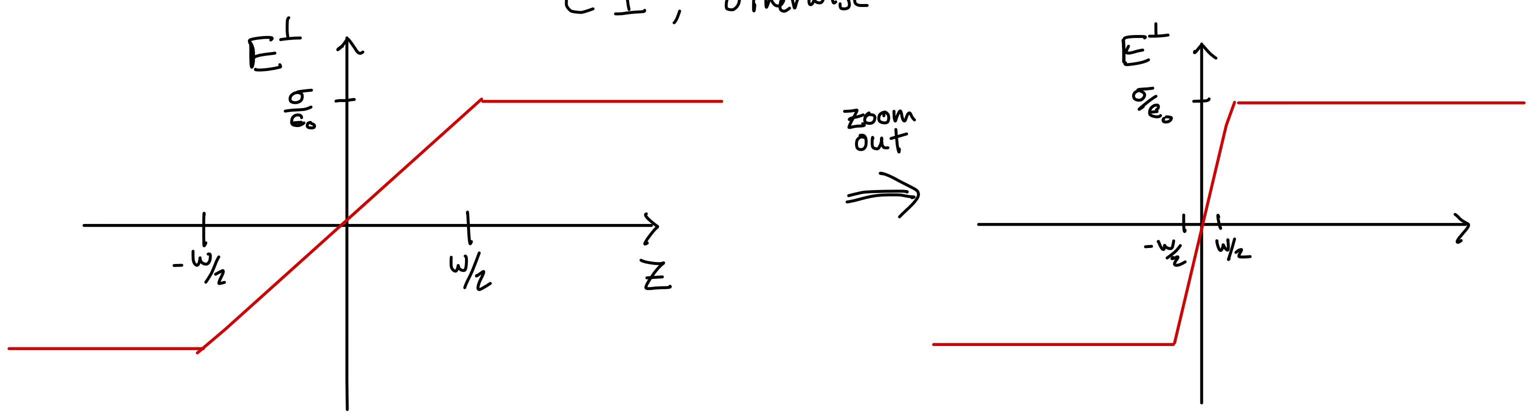
$$\rho(x,y,z) = \begin{cases} \dfrac{\sigma(x,y)}{w}, & |z| < \frac{w}{2} \\ 0 & \text{otherwise} \end{cases}$$



Then, using Gauss law the field is

$$\int \vec{E} \cdot d\vec{a} = \frac{Q_{encl}}{\epsilon_0} = \frac{\sigma A}{\epsilon_0} \times \begin{cases} \dfrac{2z}{w}, & |z| < w/2 \\ 1, & \text{otherwise} \end{cases}$$

$$E^{\perp}(z) = \frac{\sigma}{\epsilon_0} \times \begin{cases} 2z/w, & |z| < w/2 \\ 1, & \text{otherwise} \end{cases}$$



So if w is thin enough, it looks like $E^{\perp}$ is discontinuous.

Modeling real materials

Limiting cases — perfect insulator and perfect conductor.

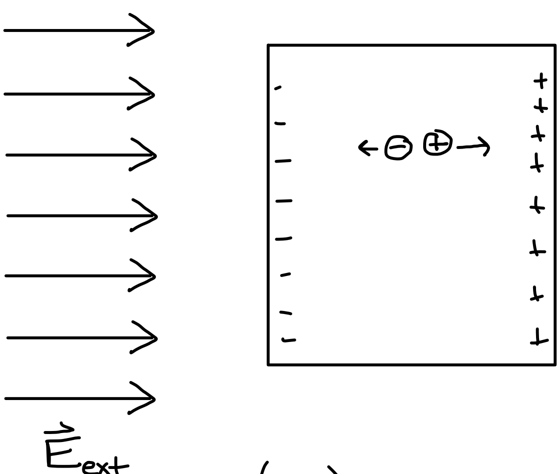
Perfect insulator: • charges are immobile - electrons are stuck onto atoms

Perfect conductor: • Charges move freely throughout the material — electrons are delocalized.
• Unlimited supply of charge

Today we'll discuss conductors.

Properties of conductors:

(i) $\vec{E} = 0$ inside a conductor (for electrostatics)



Since the charge is free to move, and $\vec{F} = q\vec{E}$, it will move until $\vec{E}=0$.

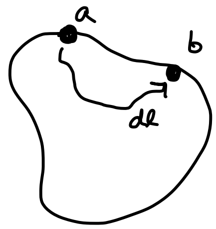
(ii) $\rho = 0$ inside a conductor
$\vec{\nabla}\cdot\vec{E} = \rho/\epsilon_0$, $\vec{E}=0 \Rightarrow \rho = 0$.

(iii) Any net charge lives on the surface

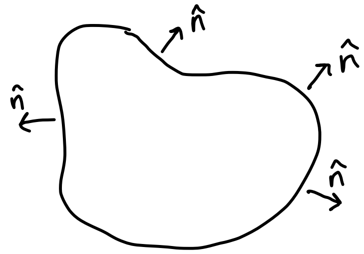
(iv) The conductor is an equipotential:

$$V_b - V_a = -\int_a^b \vec{E}\cdot d\vec{\ell} = 0.$$

Note that this applies even if a and/or b are on the surface:

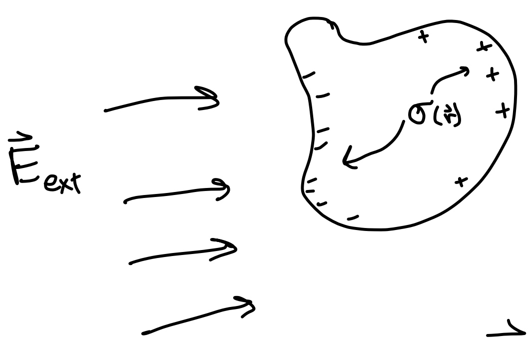


v) Just outside a conductor, $\vec{E}$ points in the normal / perpendicular direction $\hat{n}$.



If there were a parallel component, charge would flow along the surface.

Example: Chunk of conductor placed in an external field.



Charge moves so that there is non-zero $\sigma(\vec{r})$. What is $\vec{E}$ just outside the surface?

Recall our boundary condition

$$\vec{E}_{above} - \vec{E}_{below} = \frac{\sigma}{\epsilon_0}\hat{n}$$

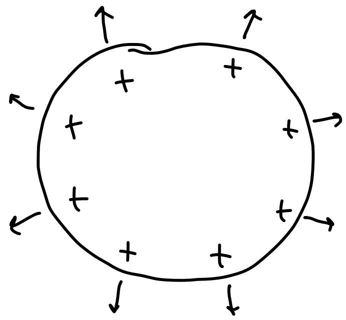
$\vec{E}_{below}$ is inside the conductor, so it's 0.

so

$$\vec{E}_{above}(\vec{r}) = \frac{\sigma(\vec{r})}{\epsilon_0}\hat{n}$$

(Different from a sheet of charge)

Force on a charged conductor. The charges want to fly off to ∞ but are constrained to the conductor. Newton's 3rd law says there must be a reaction force on the conductor. What is it?



On the surface, $\vec{E}_{above} = \frac{\sigma}{\epsilon_0}\hat{n}$ and $\vec{E}_{below} = 0$.

You might be tempted to say the force per unit area is $\frac{\vec{F}}{A} = \sigma\vec{E} = \frac{\sigma^2}{\epsilon_0}\hat{n}$ [not quite!]

and this is almost right. We can get the right answer a couple of ways

Energy: Take a small patch and push it outward (in the $\hat{n}$ direction) a tiny bit.

The work done is $W = -F\,d\ell$. But we've also set the field to zero inside a volume $A\,d\ell$.

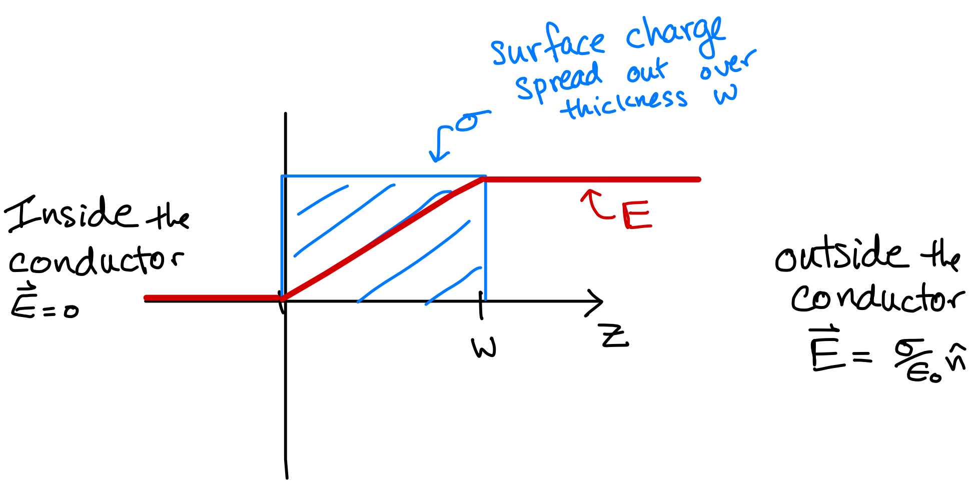
so

$$-F\cdot d\ell = -A\,d\ell \cdot \left(\frac{\epsilon_0}{2}E^2\right) = -A\,d\ell \cdot \frac{\epsilon_0}{2}\left(\frac{\sigma}{\epsilon_0}\right)^2$$

$$\vec{F}/A = \frac{\sigma^2}{2\epsilon_0}\hat{n} \qquad \text{[correct]}$$

Note the factor of 2.

Forces: Alternatively, we can remember that an infinitely-thin surface charge distribution is an idealization If it's actually spread over some finite thickness, the field isn't discontinuous. It interpolates between 0 and $\frac{\sigma}{\epsilon_0}$.



The charge density and field are

$$\rho = \frac{\sigma}{w}, \qquad E = \frac{\sigma}{\epsilon_0}\left(\frac{z}{w}\right) \qquad 0 \le z \le w$$

and so

$$\frac{\vec{F}}{A} = \int \vec{E}(z)\,\rho(z)\,dz = \int_0^w \frac{\sigma}{\epsilon_0}\left(\frac{z}{w}\right)\hat{k}\,\frac{\sigma}{w}\,dw$$

$$= \frac{\sigma^2}{\epsilon_0 w^2}\cdot\frac{1}{2}w^2\,\hat{k}$$

$$= \frac{\sigma^2}{2\epsilon_0}\,\hat{k}\,.$$