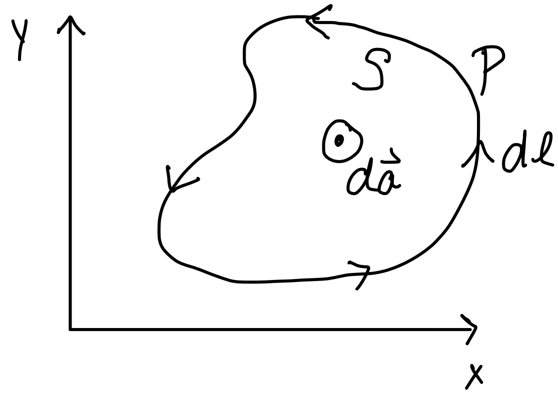
Tuesday Oct 1, 2019

Stokes' Theorem:

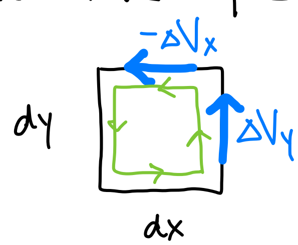
$$\int_S (\vec{\nabla}\times\vec{v})\cdot d\vec{a} = \oint_P \vec{v}\cdot d\vec{l}$$

XY plane



Right Hand Rule: $d\vec{a}$ points out of the board.
(This follows from the use of the RHR in defining $\vec{\nabla}\times\vec{v}$.)

Intuitive picture: Curl in small patch



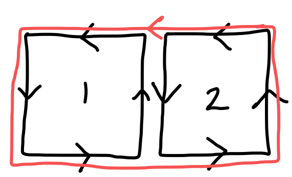
$$\vec{\nabla}\times\vec{V} = \left(\frac{\partial V_y}{\partial x} - \frac{\partial V_x}{\partial y}\right)\hat{z} \approx \frac{\Delta V_y}{\Delta x} - \frac{\Delta V_x}{\Delta y}$$

$$\Delta\vec{a} = \Delta x \Delta y\,\hat{z}$$

take $V_y(0)=0$, $V_x(0)=0$ $\Rightarrow$ $\int_S (\vec{\nabla}\times\vec{v})\cdot d\vec{a} \approx \left(\frac{\Delta V_y}{\Delta x} - \frac{\Delta V_x}{\Delta y}\right)\hat{z}\cdot(\Delta x\Delta y\,\hat{z}) = \Delta V_y \Delta y - \Delta V_x \Delta x$

$$\oint_P \vec{v}\cdot d\vec{l} = V_x(0)\Delta x + (V_y(0)+\Delta V_y)\Delta y + (V_x(0)+\Delta V_x)(-\Delta x) + V_y(0)(-\Delta y)$$
$$= \Delta V_y \Delta y - \Delta V_x \Delta x$$

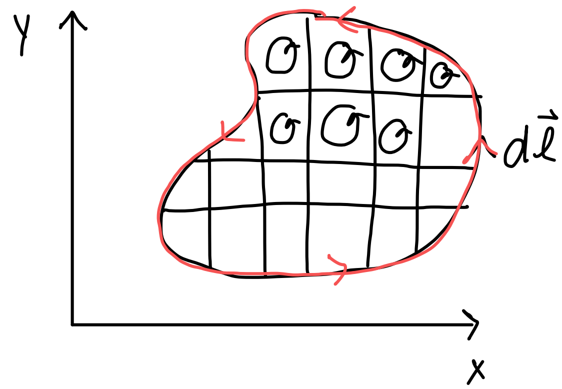
Two rectangular patches:



area: $a_{tot} = a_1 + a_2$

Integral around perimeter: $\oint_{tot} d\vec{l} = \oint_1 d\vec{l} + \oint_2 d\vec{l}$ ← Part along the shared border cancels.

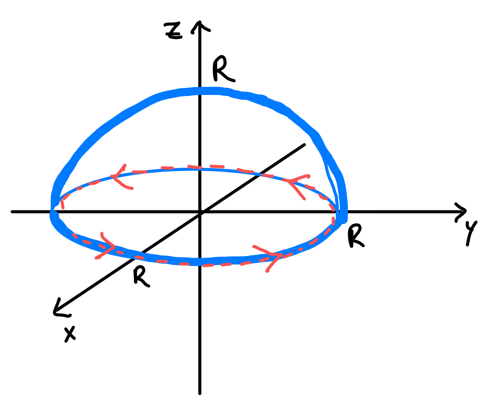
We can take any area and chop it up into tiny patches. All the internal lines are shared by two patches and cancel, so we are left with the perimeter.



Example: $\vec{V}(x,y,z) = 4y\,\hat{i} + x\,\hat{j} + 2z\,\hat{k}$

Compute $\int (\vec{\nabla}\times\vec{V})\cdot d\vec{a}$ over the hemisphere

$x^2+y^2+z^2=R^2$, $z \geq 0$. We'll do this 3 ways.



1) Direct evaluation:

$$\vec{\nabla}\times\vec{V} = \det\begin{vmatrix} \partial_x & \partial_y & \partial_z \\ V_x & V_y & V_z \\ \hat{i} & \hat{j} & \hat{k} \end{vmatrix}$$

$$= \left(\frac{\partial V_z}{\partial y} - \frac{\partial V_y}{\partial z}\right)\hat{i} + \left(\frac{\partial V_x}{\partial z} - \frac{\partial V_z}{\partial x}\right)\hat{j} + \left(\frac{\partial V_y}{\partial x} - \frac{\partial V_x}{\partial y}\right)\hat{k}$$

$$= 0\,\hat{i} + 0\,\hat{j} + (1-4)\hat{k} = -3\hat{k}$$

$$d\vec{a} = R^2 \sin\theta\, d\theta\, d\phi\, \hat{r} \quad \Rightarrow \quad \hat{k}\cdot\hat{r} = \cos\theta$$

$$\int(\vec{\nabla}\times\vec{V})\,d\vec{a} = \int_0^{2\pi} d\phi \int_0^{\pi/2} (-3)R^2 \sin\theta\cos\theta\, d\theta = -6\pi R^2 \int_0^1 u\,du = -3\pi R^2.$$

$u = \sin\theta$

2) Stokes' theorem $\int_S (\vec\nabla\times\vec v)\cdot d\vec a = \oint_P \vec v\cdot d\vec l$

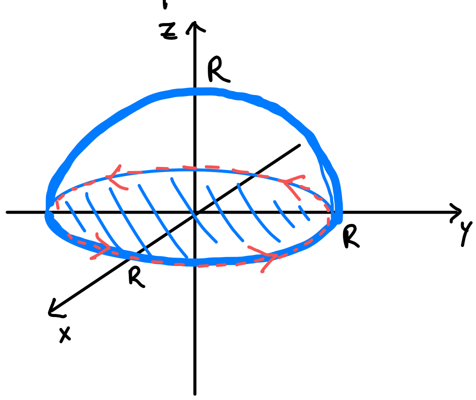
The boundary is a circle $x^2+y^2=R^2$, $z=0$.
Which way around? Right hand rule $\rightarrow$ counter clockwise.

$$d\vec l = R\,d\phi\,\hat\phi = R\,d\phi\,(\cos\phi\,\hat j - \sin\phi\,\hat\imath)$$

$$\vec v\cdot d\vec l = R\,d\phi\,(x\cos\phi - 4y\sin\phi) \qquad x = R\cos\phi,\ y = R\sin\phi$$
$$= R\,d\phi\,(R\cos^2\phi - 4R\sin^2\phi)$$

$$\oint_P \vec v\cdot d\vec l = \int_0^{2\pi} R^2(\cos^2\phi - 4\sin^2\phi)\,d\phi$$
$$= R^2\cdot(\pi - 4\pi) = -3\pi R^2$$

3) Use Stokes' theorem to relate integral over the hemisphere to an integral in the plane with the same boundary.



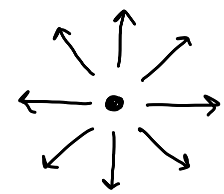
Now $d\vec a = dx\,dy\,\hat k$ (sign from RHR)

$$\vec\nabla\times\vec v = -3\hat k \Rightarrow (\vec\nabla\times\vec v)\cdot d\vec a = -3\,dx\,dy$$

$$\int_{S'}(\vec\nabla\times\vec v)\cdot d\vec a = -3\underbrace{\int_{S'} dx\,dy}_{\text{Area of circle with radius } R} = -3\pi R^2$$

Dirac δ function

Suppose $\quad \vec{v}(\vec{r}) = \frac{1}{r^2}\hat{r} = \frac{\vec{r}}{r^3}$



What is the divergence of $\vec{v}$?
Using the formula for $\vec{\nabla}\cdot\vec{v}$ in spherical coordinates:

$$\vec{\nabla}\cdot\vec{v} = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 v_r\right) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}\left(v_\theta\sin\theta\right) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\phi}v_\phi$$

$$= \frac{1}{r^2}\frac{\partial}{\partial r}(1) + 0 + 0 = 0$$

This is strange, because $\vec{v}(\vec{r})$ definitely looks "divergencey". What's going on?

Let's try with the divergence theorem over some sphere of radius R centered on $r=0$.

$$\int(\vec{\nabla}\cdot\vec{v})\,d^3r = \oint_S \vec{v}\cdot d\vec{a} \qquad d\vec{a} = R^2\sin\theta\, d\theta\, d\phi\,\hat{r}$$

$$= \iint \frac{1}{R^2}R^2\sin\theta\, d\theta\, d\phi$$

$$= 4\pi \qquad \text{independent of } R \ (!?!)$$

The source of the trouble is that all of the divergence in $\vec{v}(\vec{r})$ is located at $r=0$. The formula for $\vec{\nabla}\cdot\vec{v}$ in spherical coordinates involves $\frac{1}{r^2}$ and $\frac{1}{r\sin\theta}$, so it <u>cannot</u> be valid at $r=0$.

We also notice that $|\vec{v}(\vec{r})| \to \infty$ as $r \to 0$

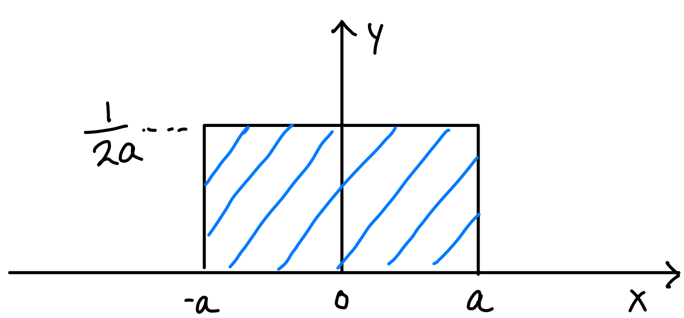
To summarize, $\vec{\nabla}\cdot\vec{v}$ is zero everywhere except for at $r=0$, where it is ∞. But it is ∞ in such a way that integrating over it gives a finite value, namely 4π.

This is a strange situation, but it is very important in E&M when dealing with point charges, so we should figure out how to deal with it.
These properties are described by the Dirac delta function.

Dirac δ in 1D:

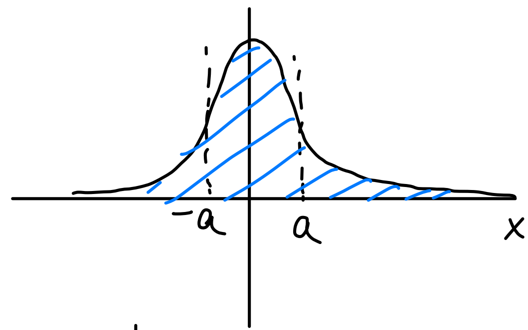
$$\delta(x) = \begin{cases} 0 & \text{if } x \neq 0 \\ \infty & \text{if } x = 0 \end{cases}, \qquad \int_a^b \delta(x)\,dx = \begin{cases} 1 & \text{if } a<0<b \\ 0 & \text{otherwise} \end{cases}$$

As a limit: $$\delta(x) = \lim_{a\to 0} \begin{cases} \frac{1}{2a} & -a<x<a \\ 0 & \text{otherwise} \end{cases}$$



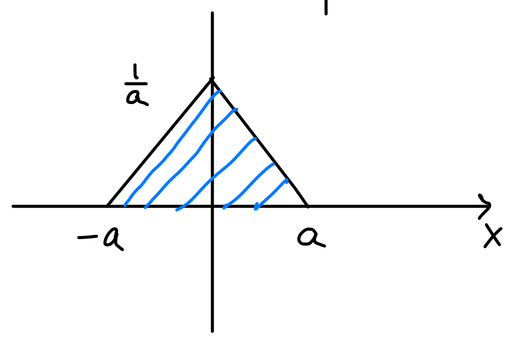
We could also take the limit of e.g. a Gaussian:

$$\delta(x) = \lim_{a \to 0} \frac{1}{a\sqrt{\pi}} e^{-x^2/a^2}$$



or a triangle

$$\delta(x) = \lim_{a \to 0} \begin{cases} \frac{1}{a}\left(1 - \frac{x}{|a|}\right) & -a < x < a \\ 0 & \text{otherwise} \end{cases}$$

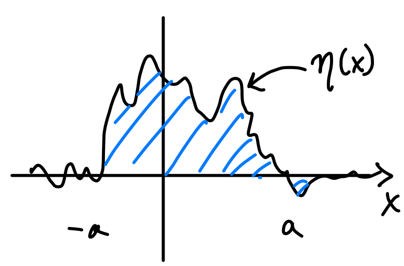


Really, a δ function can be constructed by starting with any function $\eta(x)$ which has a finite integral over all x (which we can normalize to 1)

$$\int_{-\infty}^{\infty} \eta(x)\, dx = 1$$

and then we rescale by a factor $a > 0$ so that

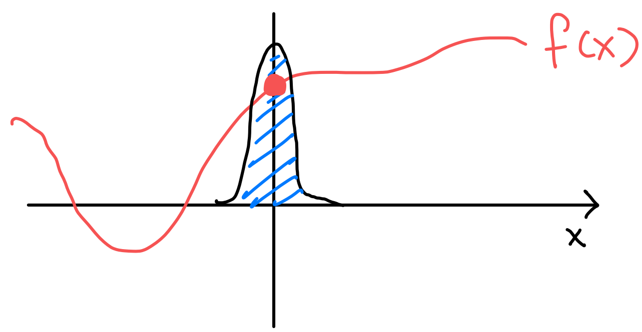
$$\eta_a(x) = \frac{1}{a} \eta(x/a), \qquad \delta(x) = \lim_{a \to 0} \eta_a(x).$$



"Sifting" property: Consider some "normal" (i.e. finite) function $f(x)$

$$\boxed{\int_{-\infty}^{\infty} \delta(x)\, f(x)\, dx = f(0)}$$

This is the most important feature of a δ function. Note that the integral need not run from $-\infty$ to ∞. It just needs to include 0.



To understand this property, let's use the limit of the rectangle step function, which allows us to limit the integral $\int_{-a}^{a}$

$$\int_{-\infty}^{\infty} \delta(x)\, f(x)\, dx = \lim_{a \to 0} \int_{-a}^{a} \frac{1}{2a} f(x)\, dx$$

and expand $f(x)$ as a Taylor series about $x=0$:

$$f(x) = f(0) + f'(0)\,x + \frac{1}{2} f''(0)\,x^2 + \frac{1}{3!} f'''(0)\, x^3 + \ldots$$

so

$$\int_{-\infty}^{\infty} \delta(x)\, f(x)\, dx = \lim_{a \to 0} \frac{1}{2a} \left[\int_{-a}^{a} f(0)\, dx + \int_{-a}^{a} f'(0)\, x\, dx + \int_{-a}^{a} \frac{1}{2} f''(0)\, x^2\, dx + \ldots \right]$$

Now $\int_{-a}^{a} dx = 2a$, $\int_{-a}^{a} x\, dx = \frac{1}{2}(a^2 - a^2) = 0$, $\int_{-a}^{a} x^2\, dx = \frac{1}{3}(a^3 - (-a)^3) = \frac{2}{3} a^3$

So $$\int_{-\infty}^{\infty} \delta(x) f(x)\, dx = \lim_{a\to 0} \frac{1}{2a}\left[f(0)\cdot 2a + f'(0)\cdot 0 + f''(0)\cdot \frac{2}{3}a^3 + \dots \right]$$

$$= \lim_{a\to 0}\left[f(0) + f''(0)\, a^2 \cdot \frac{1}{3} + \dots \right]$$

In general, each term in the series has the form $f^{(n)}(0)\, a^n \cdot$[finite number], and all terms aside from $f(0)$ disappear in the limit $a \to 0$, and

$$\int_{-\infty}^{\infty} \delta(x) f(x)\, dx = f(0).$$

(Note that this only holds if the derivatives of f are finite at $x=0$.)

In practice, we don't need a to literally be zero. We just need it so small that $f^{(n)}(0)\, a^n$ is negligible compared to, say, the precision of a measurement we can make. This means that the function $f(x)$ should not change appreciably over a length scale a.

Coming back to electrodynamics, if we have some charge distribution $\rho(x)$ which is localized within some length scale a, and we place it in an electric field $\vec{E}(x)$ which varies on a length scale $L \gg a$, then we will not make much error by assuming that all of the charge is located at a single point:

$$\rho(x) \to q\,\delta(x), \quad q = \int_{-\infty}^{\infty} \rho(x)\, dx.$$

Now, back to the properties of the δ function.
We can shift the origin: $\delta(x-x_0)$ then

$$\int_{-\infty}^{\infty} \delta(x-x_0)\, f(x)\, dx = f(x_0)$$

Units: Since $\int_{-\infty}^{\infty} \delta(x)\, dx = 1$, dx has units of meters, so $\delta(x)$ has units of density, i.e. $\frac{1}{m}$.

Scaling: $\delta(kx) = \frac{1}{|k|}\delta(x)$ see Griffiths Ex. 1.15.

3D δ function:

$$\delta^{(3)}(\vec{r}) = \delta(x)\delta(y)\delta(z) \Leftarrow \text{units are } \frac{1}{m^3}$$

$$\int d^3r\, \delta^{(3)}(\vec{r}) = 1$$

$$\int d^3r\, \delta^{(3)}(\vec{r}-\vec{r}_0)\, f(\vec{r}) = f(\vec{r}_0)$$

Notation: Griffiths uses $\delta^{(3)}(\vec{r})$ for a 3D δ. Other authors use $\delta(\vec{r})$. Either way is fine, and ambiguity should not arise because if the argument is a vector, each element of the vector must be zero.

Finally, back to $\vec{\nabla}\cdot\vec{v}$. We found that

$$\vec{\nabla}\cdot\vec{v} = \begin{cases} \infty & \vec{r}=0 \\ 0 & \text{otherwise} \end{cases} \qquad \text{and} \qquad \int d^3r\, \vec{\nabla}\cdot\vec{v} = 4\pi$$

So this can be explained by setting

$$\vec{\nabla}\cdot\left(\frac{\hat{r}}{r^2}\right) = 4\pi\,\delta^{(3)}(\vec{r}).$$

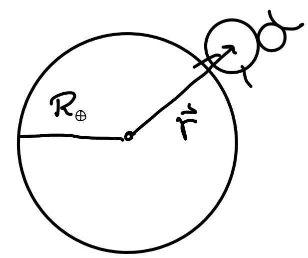
Also, since $\vec{\nabla}\left(\frac{1}{r}\right) = -\frac{\hat{r}}{r^2}$, we have

$$\boxed{\vec{\nabla}\cdot\vec{\nabla}\left(\frac{1}{r}\right) = \nabla^2\left(\frac{1}{r}\right) = -4\pi\,\delta^{(3)}(\vec{r})}$$

[Bonus materal, time permitting]

When setting $\rho(\vec{r}) \propto \delta(\vec{r})$ works: gravity at Earths surface

Force on a spherical cow: $\vec{F} = \int \underbrace{\vec{g}(\vec{r})}_{\text{gravitational field at } \vec{r}}\, \rho_{mass}(\vec{r})\, d^3r$



Take $\rho_{mass}(\vec{r}) \longrightarrow M_{cow}\,\delta^{(3)}(\vec{r}_{cow})$

$$\vec{F} = M_{cow}\,\vec{g}(\vec{r}_{cow}) \qquad\qquad \vec{g}(\vec{r}) = \frac{-GM_\oplus}{r^2}\,\hat{r}$$

What error have we made? Cow size is $a \sim 1\text{m}$. Leading correction

$$\sim \frac{\partial g}{\partial r} a \sim \left.\frac{GM_\oplus}{r^3}\right|_{r=R_\oplus} a \approx g(R_\oplus)\frac{a}{R_\oplus}.$$ Correction is $\sim \frac{1\text{m}}{6\times10^6\text{m}}$ (small).

[Since G is only known to 5 sig figs, this is fine.]

When $\rho(\vec{r}) \propto \delta^{(3)}(\vec{r})$ <u>doesn't</u> work:

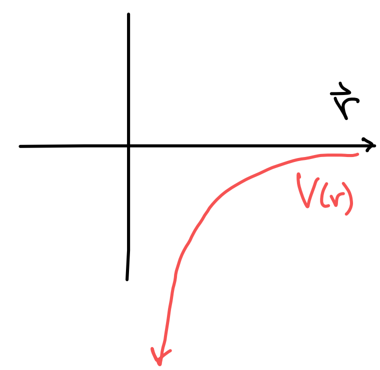
Self-energy of a point charge. (This will be covered later, but hopefully you can believe for now) the energy to construct a charge distribution $\rho(\vec{r})$ is

$$W = \frac{1}{2}\int_{\text{all space}} \rho(\vec{r})\, V(\vec{r})\, d^3r$$

here $V(\vec{r})$ is the electrostatic potential. For a point charge, $\rho(\vec{r}) \to q\,\delta^{(3)}(\vec{r})$, $V(\vec{r}) = \frac{q}{4\pi\epsilon_0 r}$

so

$$W \to \frac{1}{2}\, q \cdot \frac{q}{4\pi\epsilon_0 (0)} = \infty. \text{ Uh oh.}$$



The source of the trouble is evident if we go back and try to Taylor expand $V(r)$ about $r=0$:

$$V(0) = \lim_{r\to 0} \frac{q}{4\pi\epsilon_0}\left(\frac{1}{r} - \frac{1}{r^2} + \frac{1}{r^3} - \frac{1}{r^4} + \ldots\right)$$

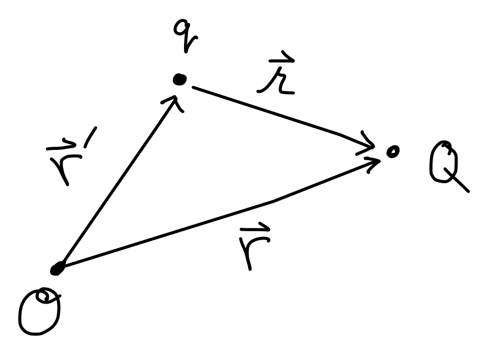
Every term blows up at $r \to 0$, so the requirement that $f^{(n)}(0)\, a^n$ go to zero doesn't hold. $V(r)$ is not approximately constant over <u>any</u> length scale around $r=0$ and so the replacement $\rho(\vec{r}) \to q\,\delta^{(3)}(\vec{r})$ doesn't work.

Another way to look at it is that $V(\vec{r})$ depends on the details of $\rho(\vec{r})$ in a way that doesn't vanish when the size of $\rho(\vec{r})$ shrinks to zero.

Thursday Oct 3, 2019

Electrostatics

Coulomb's Law



Force on "test" charge Q at $\vec{r}$ due to a "source" charge q at $\vec{r}\,'$

$$\vec{F} = \frac{qQ}{4\pi\epsilon_0 |\vec{r}-\vec{r}\,'|^2} \frac{\vec{r}-\vec{r}\,'}{|\vec{r}-\vec{r}\,'|} = \frac{qQ}{4\pi\epsilon_0 \mathscr{r}^2}\hat{\mathscr{r}}$$

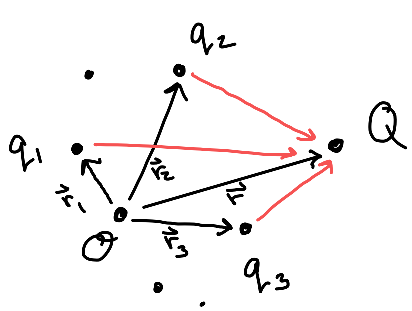
where $\vec{\mathscr{r}} \equiv \vec{r}-\vec{r}\,'$ points from source to observer.

$\epsilon_0 = 8.85 \times 10^{-12} \frac{C^2}{Nm^2}$ "permittivity of free space"

electron charge: $e = -1.6 \times 10^{-19}\, C$

$\rightarrow$ Force between 2 electrons @ 1m: $\approx 2 \times 10^{-28}$ N.

Superposition:



Force on test charge Q due to all source charges q_1 q_2 q_3 ... is just the sum of the individual forces

$$\vec{F} = \vec{F}_1 + \vec{F}_2 + \vec{F}_3 + \ldots$$

$$= \frac{Q}{4\pi\epsilon_0}\left[\frac{q_1}{|\vec{r}-\vec{r}_1|^3}(\vec{r}-\vec{r}_1) + \frac{q_2}{|\vec{r}-\vec{r}_2|^3}(\vec{r}-\vec{r}_2) + \ldots\right]$$

$$\equiv Q\vec{E}(\vec{r})$$

Electric field: $\vec{E}(\vec{r}) = \frac{1}{4\pi\epsilon_0} \sum_i \frac{q_i}{|\vec{r}-\vec{r}_i|^3} (\vec{r}-\vec{r}_i)$

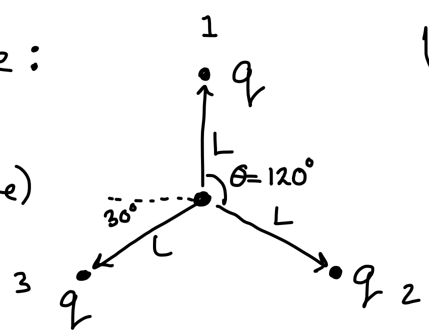
Force per unit charge

What is a field?

Without bothering too much about philosophy, a field is a quantity (like a number, or a vector - i.e. 3 numbers) assigned to every point in space.

Example:

(in x-y plane)



What is $\vec{E}$? Ask students.

Need the vector sum of the field from each charge.

$$\vec{E}_1 = \frac{q}{4\pi\epsilon_0 L^2}(-\hat{j})$$

$$\vec{E}_2 = \frac{q}{4\pi\epsilon_0 L^2}\left(\cos(30°)\,\hat{i} + \sin(30°)\,\hat{j}\right)$$

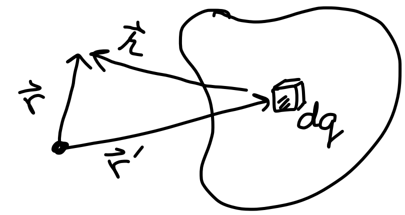
$$E_3 = \frac{q}{4\pi\epsilon_0 L^2}\left(-\cos(30°)\,\hat{i} + \sin(30°)\,\hat{j}\right)$$

$$\vec{E} = \vec{E}_1 + \vec{E}_2 + \vec{E}_3 = \frac{q}{4\pi\epsilon_0 L^2}\left[\left(0+\cos(30°)-\cos(30°)\right)\hat{i} + \left(-1+\sin(30°)+\sin(30°)\right)\hat{j}\right]$$

$$= 0$$

As we could have guessed from symmetry arguments.

Continuous charge distribution:



$$\sum_i q_i \rightarrow \int dq$$

"dq" means "tiny bit of charge"

$$\vec{E}(\vec{r}) = \frac{1}{4\pi\epsilon_0}\int dq \, \frac{\vec{r}-\vec{r}\,'}{|\vec{r}-\vec{r}\,'|^3} = \frac{1}{4\pi\epsilon_0}\int dq \, \frac{\hat{\mathscr{r}}}{\mathscr{r}^2}$$

This is 3 integrals:

$$E_x(\vec{r}) = \frac{1}{4\pi\epsilon_0}\int dq \, \frac{\hat{\mathscr{r}}\cdot\hat{i}}{\mathscr{r}^2}$$ and similar $E_y(\vec{r})$ $E_z(\vec{r})$.

Forms for dq

Volume charge density: $dq = \rho(\vec{r})\, d^3r = \rho(\vec{r})\, d\tau$

Griffiths' notation

Brief aside:
What happens at $\vec{r} = \vec{r}\,'$, i.e. $\vec{\mathscr{r}} = 0$? Does it blow up?
To simplify, choose origin so that $\vec{r} = 0$, $\vec{\mathscr{r}} = -\vec{r}\,'$.

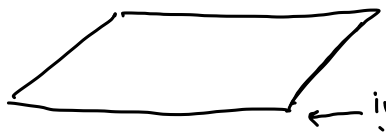
$$\vec{E}(\vec{r}=0) = \frac{1}{4\pi\epsilon_0}\int \rho(\vec{r}\,')\frac{(-\hat{r}')}{r'^2}\, r'^2 \sin\theta' \, dr' \, d\theta' \, d\phi'$$

powers of r cancel, no disaster

Surface charge distribution 2D

$$dq = \sigma \, da$$

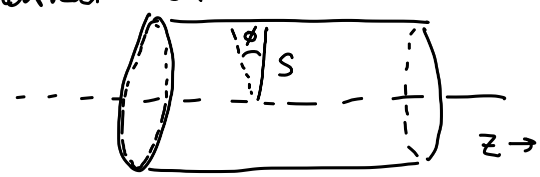
$\sigma =$ surface charge density
units : $\frac{C}{m^2}$



infinitessimally thin in z direction

$$\rho(\vec{r}) = \sigma(x,y)\,\delta(z)$$

cylindrical shell

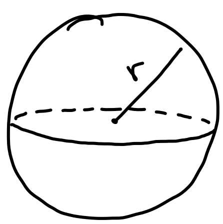


$$\rho(\vec{r}) = \sigma(\phi,z)\,\delta(s)$$

(infinitessimally thin in s direction)

Griffiths' notation
s is radius in cylindrical coordinates.

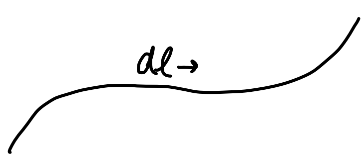
spherical shell



$$\rho(\vec{r}) = \sigma(\theta,\phi)\,\delta(r)$$

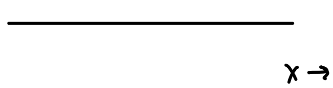
(infinitessimally thin in r direction)

Linear charge distribution 1D



$$dq = \lambda \, d\ell$$

length along the line



$$\rho(\vec{r}) = \lambda(x)\,\delta(y)\,\delta(z)$$

Point charge distribution 0D

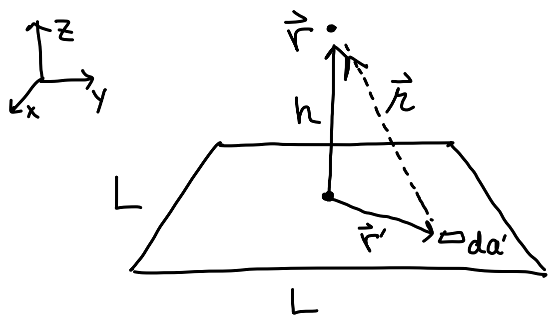
$$dq \rightarrow q$$

q •

$$\rho(\vec{r}) = q\,\delta(\vec{r}) = q\,\delta(x)\,\delta(y)\,\delta(z)$$

Example: uniformly charged square

$\sigma = Q/L^2$



Before we start, should $\vec{E}$ have a horizontal component? If it did, which way would it point?

$\vec{r} = (0,0,h)$ $\quad dq = \sigma\, dx dy$

$\mathscr{r}^2 = x^2+y^2+h^2, \quad \vec{\mathscr{r}} = -x\hat{i} - y\hat{j} + h\hat{k}$

Do the integral for E_z, since $E_x = E_y = 0$.

$$E_z(\vec{r}) = \frac{1}{4\pi\epsilon_0}\int dq \frac{\hat{\mathscr{r}}\cdot\hat{k}}{\mathscr{r}^2} = \frac{1}{4\pi\epsilon_0}\int_{-L/2}^{L/2}\int_{-L/2}^{L/2} \sigma\, dx dy \frac{h}{[x^2+y^2+h^2]^{3/2}}$$

Not a particularly pleasant integral. Before pushing on, let's look at some limiting cases.

Limit $z \gg L \Rightarrow x^2+y^2+z^2 \approx z^2 \Longrightarrow E_z \approx \frac{1}{4\pi\epsilon_0}\frac{1}{z^2}\cdot \sigma L^2 = \frac{Q}{4\pi\epsilon_0 z^2}$

Limit $z \ll L \Rightarrow x^2+y^2+z^2 \approx x^2+y^2 \Rightarrow E_z \approx \frac{\sigma z}{4\pi\epsilon_0}\iint \frac{dx dy}{[x^2+y^2]^{3/2}}$

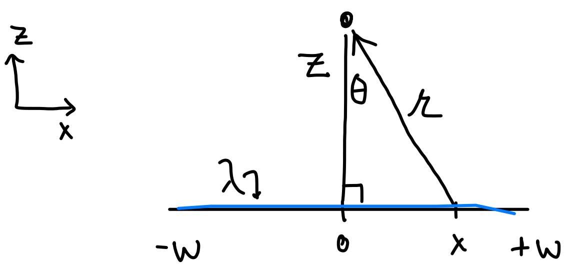
blows up @ $x=y=0$. uh oh...

What went wrong? In the neighborhood of $x=y=0$, we can't make the approximation $x^2+y^2+z^2 \approx x^2+y^2$.

Let's just calculate it. To do this, we'll actually solve two easier problems and build from there.

Step 1

First, find the field at height z above the center of a line of charge of length $2w$.



$$dq = \lambda dx \qquad \vec{r} = -x\hat{i} + z\hat{k}$$

By symmetry, we only need to worry about the $\hat{k}$ component of $\vec{E}$.

$$\vec{E}_{line} = E_{line}\hat{k}$$

$$\hat{r}\cdot\hat{k} = \cos\theta, \quad r = \frac{z}{\cos\theta}$$

$$x = z\tan\theta, \quad dx = z\frac{1}{\cos^2\theta}d\theta$$

$$E_{line}(z) = \frac{1}{4\pi\epsilon_0}\int\frac{dq}{r^2}\hat{r}\cdot\hat{k} = \frac{1}{4\pi\epsilon_0}\int_{\theta_{-w}}^{\theta_w}\frac{\lambda z d\theta}{\cos^2\theta}\frac{\cos^2\theta}{z^2}\cos\theta$$

$$= \frac{1}{4\pi\epsilon_0}\lambda\frac{1}{z}\int_{\theta_{-w}}^{\theta_w}\cos\theta d\theta$$

$$= \frac{1}{4\pi\epsilon_0}\frac{\lambda}{z}\left(\sin\theta_w - \sin\theta_{-w}\right)$$

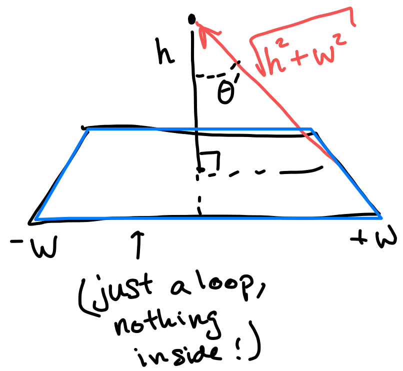
$\theta_w \equiv \theta$ when $x = w$

$\sin\theta = \frac{w}{r}$

$$\vec{E}_{line} = \frac{2\lambda}{4\pi\epsilon_0 z}\frac{w}{\sqrt{w^2+z^2}}\hat{k}.$$

Step 2.

Next, we'll compute the field due to a square loop of uniformly charged wire, with side $2w$

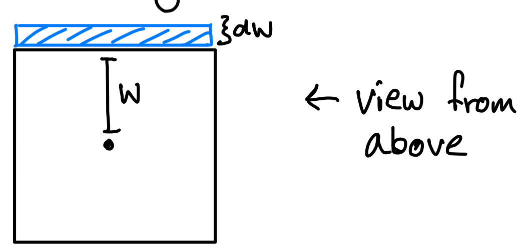


This is almost just 4 copies of the line. The only difference is that they each generate a field pointing a different way. We only want the $\hat{k}$ component, which introduces a new angle $\vec{E}_{line} \cdot \hat{k} = E_{line} \cos\theta'$.
The distance is also now $\sqrt{h^2+w^2}$, rather than just z.

So we sub $z = \sqrt{h^2+w^2}$, multiply by $\cos\theta' = \frac{h}{\sqrt{h^2+w^2}}$ and multiply by 4 for the four segments:

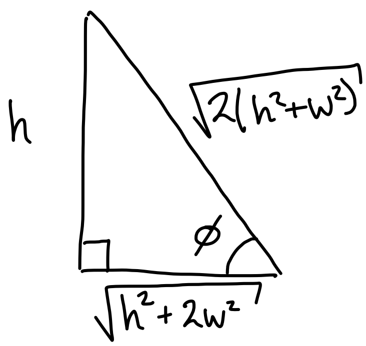
$$E_{loop}(h,w) = \frac{2\lambda}{4\pi\epsilon_0} \frac{4w}{h^2+w^2} \cdot \frac{h}{\sqrt{h^2+2w^2}}.$$

Finally, we return to the square. We build it up by a bunch of loops of side $2w$, with charge density $\lambda = \sigma \cdot dw$, and integrate w from 0 to $\frac{L}{2}$.



$$E_{square} = \int_0^{L/2} \frac{2\sigma \cdot dw \cdot w\, h}{\pi\epsilon_0 (h^2+w^2)\sqrt{h^2+2w^2}}$$

This is an integral you can just ask Mathematica to do for you. But we can also make a clever substitution by drawing another triangle (I'm not sure of its physical significance, but it helps make a good change of variables).



$$\tan\phi = \frac{h}{\sqrt{h^2+2w^2}}$$

$$\frac{1}{\cos^2\phi}\,d\phi = \frac{-h}{2}\,\frac{4w\,dw}{[h^2+2w^2]^{3/2}}$$

$$\cos^2\phi = \frac{h^2+2w^2}{2(h^2+w^2)}$$

so $$d\phi = \frac{h^2+2w^2}{2(h^2+w^2)} \cdot \frac{-2hw\,dw}{[h^2+2w^2]^{3/2}} = \frac{-hw\,dw}{(h^2+w^2)\sqrt{h^2+2w^2}}$$

$$E_{square} = \int_0^{L/2} \frac{2\sigma \cdot dw \cdot wh}{\pi\epsilon_0 (h^2+w^2)\sqrt{h^2+2w^2}} = -\frac{2\sigma}{\pi\epsilon_0}\int_{\phi_0}^{\phi_{L/2}} d\phi = \frac{2\sigma}{\pi\epsilon_0}\left(\phi_0 - \phi_{L/2}\right)$$

$$\tan\phi_0 = \frac{h}{h} = 1 \Rightarrow \phi_0 = \frac{\pi}{4}, \quad \tan\phi_{L/2} = \frac{h}{\sqrt{h^2+2(L/2)^2}}$$

$$\boxed{\vec{E}_{square} = \frac{\sigma}{2\epsilon_0}\left(1 - \frac{4}{\pi}\tan^{-1}\frac{1}{\sqrt{1+L^2/2h^2}}\right)\hat{k}}$$

Now taking the limit $h \ll L$ is easy $\tan^{-1} 0 = 0$

so $\vec{E}_{square} \rightarrow \frac{\sigma}{2\epsilon_0}\hat{k}$.